Regularized Theta Lifts of Harmonic Maass Forms

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

Dissertation

von

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aus Solingen

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Tag der Einreichung:	20. Oktober 2017
Tag der mündlichen Prüfung:	8. Januar 2018

Darmstadt 2018
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Danksagung

Zuerst möchte ich meinem Doktorvater Jan Hendrik Bruinier danken, ohne dessen Unterstützung diese Arbeit nicht möglich gewesen wäre. Er hat mir die Richtung gewiesen, wenn ich Hilfe brauchte, und mir die Freiheit gelassen, meine eigenen Ideen zu entwickeln.

Ebenfalls herzlich danken möchte ich Claudia Alfes-Neumann. Zu Beginn der Promotion hat sie mir alles Wichtige über die Arbeit an der Universität gezeigt und durch die gemeinsame Forschungsarbeit habe ich die mathematischen Grundlagen erlernt, auf denen diese Arbeit beruht.

Ich danke der gesamten Arbeitsgruppe Algebra für die schöne Zeit in Darmstadt. Besonders möchte ich Julian Bitterlich, Linda Frey, Michalis Neururer und Fabian Völz für viele produktive Kaffeepausen und mathematische Diskussionen danken.

Meiner Familie und meinen Freunden danke ich für die Unterstützung in- und außerhalb der Universität, ganz besonders meiner Frau Lydia und meiner Tochter Emilia.

Ich danke Jens Funke und Nils Scheithauer für die Bereitschaft, die Arbeit zu begutachten, sowie der Forschergruppe "Symmetrie, Geometrie und Arithmetik" für die finanzielle Unterstützung.

Zusammenfassung

In dieser Dissertation werden regularisierte Thetalifts zwischen verschiedenen Räumen harmonischer Maaßformen und ihre Anwendungen untersucht. Die Arbeit besteht aus drei Hauptteilen.

Im ersten Teil untersuchen wir den sogenannten Millson Thetalift, der harmonische Maaßformen vom Gewicht $-2k$ (mit $k \in \mathbb{Z}_{\geq 0}$) zu Kongruenzuntergruppen von $\mathrm{SL}_2(\mathbb{Z})$ auf vektorwertige harmonische Maaßformen vom Gewicht $1/2 - k$ abbildet. Wir zeigen, dass die Fourierkoeffizienten des Lifts einer harmonischen Maaßform F gegeben sind durch Spuren von CM-Werten und Zykelintegralen von nicht-holomorphen Modulformen, die aus F durch Anwendung gewisser Differentialoperatoren hervorgehen, und dass der Millson Thetalift mit dem klassischen Shintani Thetalift durch den ξ-Operator in Beziehung steht. Dieser Teil basiert auf einer gemeinsamen Arbeit mit Claudia Alfes-Neumann [ANS16].

Der zweite Teil behandelt neue Anwendungen des Millson und des Kudla-Millson Thetalifts. Wir konstruieren zunächst Vervollständigungen von zwei von Ramanujans Mock Thetafunktionen als Millson Thetalift einer geeigneten schwach holomorphen modularen Funktion F und benutzen dies, um Formeln für die Koeffizienten der Mock Thetafunktionen in Termen von Spuren von CM-Werten von F herzuleiten. Außerdem erhalten wir durch den Millson und den Kudla-Millson Thetalift ξ-Urbilder unärer Thetafunktionen vom Gewicht $3/2$ und $1/2$, deren holomorphe Teile rationale Fourierkoeffizienten haben. Wir benutzen diese Urbilder auch, um Petersson Skalarprodukte von harmonischen Maaßformen vom Gewicht $1/2$ und $3/2$ mit unären Thetafunktionen zu berechnen, und erhalten dadurch Formeln und Rationalitätsresultate für die Weyl-Vektoren von Borcherds-Produkten an den Spitzen. Dieser Teil basiert auf einer gemeinsamen Arbeit mit Jan Hendrik Bruinier [BS17].

Im dritten Teil erweitern wir Borcherds' regularisierten Thetalift in Signatur $(1, 2)$ auf den vollen Raum der harmonischen Maaßformen vom Gewicht $1/2$, also Formen, deren nicht-holomorpher Teil bei ∞ linear exponentiell wachsen darf. Wir erhalten reell-analytische modulare Funktionen mit logarithmischen Singularitäten an CM-Punkten und neuen Typen von Singularitäten entlang von Geodäten in der oberen Halbebene. Außerdem benutzen wir den Thetalift, um modulare Integrale vom Gewicht 2 mit rationalen Periodenfunktionen zu konstruieren, deren Koeffizienten durch Linearkombinationen von Fourierkoeffizienten von harmonischen Maaßformen vom Gewicht $1/2$ gegeben sind.

Abstract

In this thesis we study regularized theta lifts between various spaces of harmonic Maass forms and their applications. The work consists of three main parts.

In the first part we investigate the so-called Millson theta lift, which maps harmonic Maass forms of weight $-2k$ (with $k \in \mathbb{Z}_{\geq 0}$) for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ to vector valued harmonic Maass forms of weight $1/2-k$. We show that the Fourier coefficients of the lift of a harmonic Maass form F are given by traces of CM values and cycle integrals of non-holomorphic modular forms arising from F by application of certain differential operators, and that the Millson lift is related to the classical Shintani theta lift via the ξ-operator. This part is based on joint work with Claudia Alfes-Neumann [ANS16].

The second part discusses some new applications of the Millson and the Kudla-Millson theta lifts. First we construct completions of two of Ramanujan's mock theta functions using the Millson lift of a suitable weakly holomorphic modular function F and use this to derive formulas for the coefficients of the mock theta functions in terms of traces of CM values of F. Further, we use the Millson and the Kudla-Millson theta lifts to obtain ξ-preimages of unary theta functions of weight $3/2$ and $1/2$ whose holomorphic parts have rational Fourier coefficients. We also use these preimages to compute the Petersson inner products of harmonic Maass forms of weight $1/2$ and $3/2$ with unary theta series, and thereby obtain formulas and rationality results for the Weyl vectors of Borcherds products at the cusps. This part is based on joint work with Jan Hendrik Bruinier [BS17].

In the third part we extend Borcherds' regularized theta lift in signature $(1,2)$ to the full space of harmonic Maass forms of weight $1/2$, i.e., those forms whose non-holomorphic part is allowed to grow linearly exponentially at ∞. We obtain real analytic modular functions with logarithmic singularities at CM points and new types of singularities along geodesics in the upper half-plane. Further, we use the theta lift to construct modular integrals of weight 2 with rational period functions, whose coefficients are given by linear combinations of Fourier coefficients of harmonic Maass forms of weight $1/2$.

Contents

1 Introduction

1.1 Regularized theta lifts

Over the last two decades, starting with the fundamental work of Borcherds [Bor95, Bor98], regularized theta lifts between spaces of integral and half-integral weight automorphic forms have become a powerful tool in number theory. For example, Bruinier and Funke [BF06] used the so-called Kudla-Millson theta lift from weight 0 to weight 3/2 harmonic Maass forms to give a new proof and generalizations of Zagier's [Zag02] famous result on the modularity of the generating series of traces of singular moduli. Further, in [BO13], Bruinier and Ono used a variant of the Kudla-Millson theta lift to obtain a finite algebraic formula for the partition function $p(n)$ in terms of traces of CM values of a certain non-holomorphic modular function. A similar theta lift, which maps weight 0 to weight 1/2 harmonic Maass forms (which will be called the Millson theta lift in this work), was used in [AGOR15] to prove a refinement of a theorem of [BO10b] connecting the vanishing of the central derivative of the twisted L-function of a weight 2 newform and the rationality of some coefficient of the holomorphic part of a weight 1/2 harmonic Maass form.

The present work is concerned with the study of various theta lifts between spaces of harmonic Maass forms of integral and half-integral weight, their interplay, and their applications. Roughly speaking, by a **theta lift** we mean an integral operator of the shape

$$I(F,\tau) = \int_{\Gamma\backslash\mathbb{H}} F(z)\overline{\Theta(\tau,z)}y^k \frac{dx\,dy}{y^2},$$

where $\mathbb{H} = \{z = x+iy \in \mathbb{C} : y > 0\}$ is the complex upper half-plane on which $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts by Möbius transformations, $F : \mathbb{H} \to \mathbb{C}$ is some function which transforms like a modular form of weight k for Γ, and $\Theta(\tau, z)$ is an integral kernel constructed as a **theta function**, which transforms like a modular form of weight k in z and like a modular form of some weight ℓ in τ. Here k and ℓ are usually integers or half-integers. If the theta integral $I(F,\tau)$ exists for each $\tau \in \mathbb{H}$, then it transforms like a modular form of weight ℓ, so we obtain a linear map from weight k to weight ℓ automorphic forms. Such a map is very useful in constructing new examples of modular objects from known ones, and to carry over results from one space of automorphic forms to another.

In many cases, depending on the conditions imposed on F and the properties of the chosen theta function $\Theta(\tau, z)$, the integral diverges and has to be **regularized** in a suitable way to give it a meaning. For example, a simple regularization which works in many cases of interest to us is given by

$$\int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} F(z)\overline{\Theta(\tau,z)}y^k \frac{dx\,dy}{y^2} = \lim_{T\to\infty}\int_{\mathcal{F}_T} F(z)\overline{\Theta(\tau,z)}y^k \frac{dx\,dy}{y^2},$$

where

$$\mathcal{F}_T = \{z = x+iy \in \mathbb{H} : |x| \leq 1/2,\ |z| \geq 1,\ y \leq T\}$$

is a truncated fundamental domain for the action of Γ on $\mathbb{H}$. This regularization basically prescribes the order of integration. Once the convergence of a suitable regularization has been established, one can go on to study the most basic properties of the theta lift, e.g., its analytic properties (depending on the regularity of the input F) and its Fourier expansion. It often turns out that such a lift maps eigenforms of the invariant Laplace operator to eigenforms having related eigenvalues, and that it has a Fourier expansion involving interesting arithmetic information about the input form F, such as traces of CM values of F, for example. Besides the study of a single theta lift it is interesting to study the relations between different lifts, i.e., lifts constructed from different theta functions. Some theta functions are related by differential equations which translate into relations between the corresponding theta lifts.

In this work we will mainly consider regularized theta lifts of harmonic Maass forms. Following Bruinier and Funke [BF04], a **harmonic Maass form** of weight k is a smooth function $F : \mathbb{H} \to \mathbb{C}$ which transforms like a modular form of weight k, is annihilated by the weight k Laplace operator

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

and is at most of linear exponential growth at the cusps. One of the most famous applications of harmonic Maass forms is Zwegers' [Zwe02] discovery that Ramanujan's mock theta functions can be 'completed' to harmonic Maass forms by addition of suitable non-holomorphic functions, i.e., mock theta functions are the holomorphic parts of harmonic Maass forms. More generally, it often turns out that the generating series of interesting sequences (such as Hurwitz class numbers of imaginary quadratic fields or traces of CM values of modular functions, for example) are not quite modular, but are holomorphic parts of harmonic Maass forms. By computing the Fourier expansions of regularized theta lifts of suitable input functions one can often obtain the completions of interesting generating series. For example, one can construct many of Ramanujan's mock theta functions as theta lifts, giving their completions in a unified and conceptual way.

The work starts with a chapter on the necessary preliminaries about theta functions, theta lifts and harmonic Maass forms. The Chapters 3,4 and 5 contain the main results of this thesis. We close with a short outlook on future projects and open problems related to this work. Let us briefly describe the contents of the main chapters. The results of each of the three main chapters are explained in more detail in separate introductions below.

Most of the material in the preliminaries is well known and documented in the literature. The polynomial growth estimates for the coefficients of the holomorphic parts of harmonic Maass forms whose holomorphic principal part vanishes in Section 2.3.8 are new.

In Chapter 3 we investigate the Millson theta lift. It maps harmonic Maass forms of weight $-2k$ to harmonic Maass forms of weight $1/2 - k$ (for $k \in \mathbb{Z}_{\geq 0}$), and it is related to the Shintani theta lift via the ξ-operator. Further, its Fourier coefficients are given by traces of CM values and traces of cycle integrals of automorphic forms arising from F via differential operators (see Section 1.2).

In Chapter 4 we dicuss applications of the Kudla-Millson theta lift studied by Bruinier and Funke [BF06] and the Millson lift studied in this work and in [AGOR15, Alf15]. We realize two of Ramanujan's mock theta functions as images under the Millson lift of a weakly holomorphic modular function F, thereby obtaining new formulas for the coefficients of the mock theta functions in terms of traces of CM values of F. Further, we construct ξ-preimages of unary theta functions by choosing appropriate inputs for the Kudla-Millson and the Millson theta lift, and we use these preimages to compute inner products of unary theta functions with harmonic Maass forms, yielding formulas and algebraicity results for Weyl vectors of Borcherds products (see Section 1.3).

Finally, in Chapter 5 we extend the Borcherds theta lift to general harmonic Maass forms of weight 1/2. This leads to Γ-invariant functions which are real analytic on $\mathbb{H}$ up to logarithmic singularities at CM points and certain new singularities along geodesics in the upper half-plane. As an application, we consider the derivative of the Borcherds lift to construct automorphic integrals with rational period functions of weight 2 (see Section 1.4).

We will now give some more details on the main results of this work. Throughout we use the notations $\tau = u + iv, z = x + iy \in \mathbb{H}$ and $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$.

1.2 The Millson theta lift

A famous result of Zagier [Zag02] states that the twisted traces of singular moduli, i.e. the values of the modular j-invariant at quadratic irrationalities in the upper half-plane, occur as the Fourier coefficients of weakly holomorphic modular forms of weight $1/2$ and $3/2$. For example, Zagier proved that the function

$$g_1(\tau) = q^{-1} - 2 - \sum_{D<0} \mathrm{tr}_J^+(D) q^{-D}, \qquad \mathrm{tr}_J^+(D) = \sum_{Q \in \mathcal{Q}_D^+/\Gamma} \frac{J(z_Q)}{|\overline{\Gamma}_Q|},$$

is a weakly holomorphic modular form of weight $3/2$ for $\Gamma_0(4)$. Here

$$J = j - 744 = q^{-1} + 196884q + 21493760q^2 + \dots$$

is the modular j-function without its constant Fourier coefficient, the sum in the trace runs over the (finitely many) $\Gamma = \mathrm{SL}_2(\mathbb{Z})$-classes of the set $\mathcal{Q}_D^+$ of positive definite integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ of discriminant $D = b^2 - 4ac$, $\overline{\Gamma}_Q$ denotes the (finite) stabilizer of Q in $\overline{\Gamma} = \Gamma/\{\pm 1\}$ and $z_Q \in \mathbb{H}$ is the CM point associated to Q, which is characterized by $Q(z_Q, 1) = 0$. Bruinier and Funke [BF06] showed that the generating series of the traces of singular moduli can be obtained as the image of a certain theta lift of J. Using this approach, new proofs of Zagier's results, including the modularity of generating series of twisted traces of singular moduli, and generalizations to higher weight and level have been studied in several recent works, e.g. [AE13, BO13, Alf14, AGOR15]. For example, in [AGOR15] a twisted theta lift from weight 0 to weight $1/2$ harmonic Maass forms was defined which allowed to recover Zagier's generating series of weight $1/2$ as a theta lift. Further, it turned out that this lift is closely related to the Shintani lift via the ξ-operator on harmonic Maass forms. The classical Shintani lift establishes a connection between integral and half-integral modular forms [Shi75] and is an indispensable tool in the theory of modular forms. Using this relationship between integral and half-integral weight modular forms a number of remarkable theorems were proven, for example the famous theorem of Waldspurger [Wal81, ZK81, GKZ87], which asserts that the central critical value of the twisted L-function of an even weight newform is proportional to the square of a coefficient of a half-integral weight modular form.

In her thesis [Alf15], Claudia Alfes-Neumann generalized the theta lift studied in [AGOR15] to other weights, namely to a lift from weight $-2k$ to weight $1/2 - k$ harmonic Maass forms ($k \in \mathbb{Z}_{\geq 0}$). This lift, which was called Bruinier-Funke lift in [Alf15], is constructed (at least for even k) by modifying the Millson theta lift from [AGOR15] using suitable differential operators, i.e., one first applies iterated Maass raising operators to the input F to obtain something of weight 0, then plugs this into the weight 0 Millson lift from [AGOR15], and finally applies suitable iterated Maass lowering operators to make the resulting lift harmonic again. Alfes-Neumann showed (under the hypothesis that the level N of the input form is square free) that the lift indeed maps weight $-2k$ to weight $1/2-k$ harmonic Maass forms, she computed the lift of non-holomorphic Eisenstein series and Maass-Poincaré series, and she proved that the coefficients of positive index of the holomorphic part of the lift of a harmonic Maass form F are given by twisted traces of CM values of $R_{-2k}^k F$, where R_{-2k}^k is an iterated Maass raising operator which maps automorphic forms of weight $-2k$ to forms of weight 0. At the end of her thesis, she raised the question if there could be a relation between the higher weight Bruinier-Funke lift and the Shintani lift via the ξ-operator as in the case $k=0$, and she asked for a formula for the remaining Fourier coefficients of the lift.

In joint work [ANS16], we proceeded to resolve both problems, and to generalize the lift to harmonic Maass forms for arbitrary congruence subgroups. To this end, we study a (at first glance different) generalization of the theta lift considered in [AGOR15], which we call the Millson theta lift. Our lift also maps weight $-2k$ to weight $1/2-k$ harmonic weak Maass forms, where $k \in \mathbb{Z}_{\geq 0}$. It is constructed using a 'higher weight' version of the theta function used in [AGOR15], due to [Cra15]. We call it the Millson theta function. Luckily, using the higher weight Millson theta function it is easy to show that the Millson lift is related to the Shintani lift via the ξ-operator for all $k \geq 0$ as in the weight $k=0$ case. Eventually, we prove that the Millson lift (constructed using the higher weight Millson theta function) and the Bruinier-Funke lift (constructed using the weight $k=0$ Millson theta function and differential operators) agree on harmonic Maass forms up to a constant factor.

We completely determine the Fourier expansion of the Millson lift of a harmonic weak Maass form F of weight $-2k$, and we show that the coefficients of the holomorphic part of the lift are given by twisted traces of CM values of the weight 0 form $R_{-2k}^k F$, whereas the coefficients of the non-holomorphic part are given by twisted traces of geodesic cycle integrals of the weight $2k+2$ cusp form $\xi_k F(z) = 2iy^k \overline{\frac{\partial}{\partial \bar{z}} F(z)}$.

To illustrate our results, let us simplify the setup by restricting to modular forms for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In the body of this work we also treat forms for arbitrary congruence subgroups by using the theory of vector valued modular forms for the Weil representation associated to an even lattice of signature $(1,2)$.

We let $z = x+iy \in \mathbb{H}$ and $q = e^{2\pi i z}$. Recall from [BF04] that a harmonic Maass form of weight $k \in \mathbb{Z}$ is a smooth function $F : \mathbb{H} \to \mathbb{C}$ which is invariant under the usual weight k slash operation of Γ, which is annihilated by the weight k hyperbolic Laplace operator Δ_k, and which is at most of linear exponential growth at ∞. The space of such forms is denoted by H_k. We let H_k^+ be the subspace of harmonic Maass forms F for which there is a Fourier polynomial $P_F = \sum_{n \leq 0} a^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $F - P_F$ is rapidly decreasing at ∞. Every $F \in H_k^+$ has a Fourier expansion consisting of a holomorphic part F^+ and a non-holomorphic part F^-,

$$F(z) = F^+(z) + F^-(z) = \sum_{n \gg -\infty} a^+(n) q^n + \sum_{n<0} a^-(n) \Gamma(1-k, 4\pi|n|y) q^n, \qquad (1.2.1)$$

where $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt$ is the incomplete Gamma function. Harmonic Maass forms of half-integral weight for $\Gamma_0(4)$ are defined analogously. Important tools in the theory of harmonic Maass forms are the Maass lowering and raising operators $L_k = -2iy^2\frac{\partial}{\partial \bar{z}}$ and $R_k = 2i\frac{\partial}{\partial z} + ky^{-1}$, which lower or raise the weight of a real analytic modular form by 2, as well as the surjective antilinear differential operator $\xi_k : H_k^+ \to S_{2-k}$ defined by $\xi_k F(z) = 2iy^k\overline{\frac{\partial}{\partial \bar{z}}F(z)}$.

Let $D \in \mathbb{Z}$ be a discriminant. We let $\mathcal{Q}_D$ be the set of integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = D$. The modular group Γ acts on $\mathcal{Q}_D$ from the right, with finitely many classes if $D \neq 0$. For $D < 0$ we can split $\mathcal{Q}_D = \mathcal{Q}_D^+ \sqcup \mathcal{Q}_D^-$ into the subsets of positive definite ($a > 0$) and negative definite ($a < 0$) forms. Further, for $D < 0$ the stabilizer $\overline{\Gamma}_Q$ of $Q \in \mathcal{Q}_D$ in $\overline{\Gamma} = \Gamma/\{\pm 1\}$ is finite, and for $D > 0$ the stabilizer $\overline{\Gamma}_Q$ is trivial if D is a square and infinite cyclic if D is not a square.

Let $Q = [a,b,c] \in \mathcal{Q}_D$. For $D < 0$ there is an associated CM point $z_Q \in \mathbb{H}$ defined by $Q(z_Q, 1) = 0$, while for $D > 0$ the solutions of $a|z|^2 + bx + c = 0$ define a geodesic c_Q in $\mathbb{H}$, which is equipped with a certain orientation.

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant (possibly 1), and let

$$\chi_\Delta(Q) = \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } (a,b,c,\Delta) = 1 \text{ and } Q \text{ represents } n \text{ with } (n,\Delta) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2.2}$$

be a genus character. For $D < 0$ and a Γ-invariant function F we define twisted traces of CM values of F by

$$\mathrm{tr}^+_{F,\Delta}(D) = \sum_{Q \in \mathcal{Q}^+_{|\Delta|D}/\Gamma} \chi_\Delta(Q)\frac{F(z_Q)}{|\overline{\Gamma}_Q|},$$

and for $D > 0$ and a function G transforming of weight $2k+2$ for Γ we define twisted traces of cycle integrals of G by

$$\mathrm{tr}_{G,\Delta}(D) = \sum_{Q \in \mathcal{Q}_{|\Delta|D}/\Gamma} \chi_\Delta(Q) \int_{\Gamma_Q \backslash c_Q} G(z)Q(z,1)^k dz,$$

whenever the integrals converge.

Let $F \in H^+_{-2k}$ be a harmonic Maass form. We define the Millson theta lift by

$$I^M_\Delta(F,\tau) = \int^{\mathrm{reg}}_{\Gamma\backslash\mathbb{H}} F(z)\Theta_{M,k,\Delta}(\tau,z)y^{-2k}\frac{dx\,dy}{y^2}, \tag{1.2.3}$$

where $\Theta_{M,k,\Delta}(\tau,z)$ is the twisted Millson theta function, and the integral has to be regularized in a suitable way to ensure convergence. The theta function, and thus also $I^M_\Delta(F,\tau)$, transforms like a modular form of weight $1/2 - k$ in τ. We remark that the Millson theta function for $\mathrm{SL}_2(\mathbb{Z})$, and hence the Millson theta lift, vanishes identically for trivial reasons unless $(-1)^k\Delta < 0$.

For $k \in \mathbb{Z}_{\geq 0}$ with $(-1)^k\Delta < 0$ the Δ-th Shintani lift of a cusp form $G \in S_{2k+2}$ is (in our normalization) defined by

$$I_\Delta^{Sh}(G,\tau) = -|\Delta|^{-(k+1)/2} \sum_{D>0} \mathrm{tr}_{G,\Delta}(D) q^D.$$

It is well known that the Shintani lift of $G \in S_{2k+2}$ is a cusp form of weight $k+3/2$ for $\Gamma_0(4)$ which satisfies the Kohnen plus space condition, i.e., the D-th Fourier coefficient vanishes unless $(-1)^{k+1}D \equiv 0, 1(4)$. Further, it is also given by a theta lift of G.

We are now ready to state our main result for the Millson theta lift.

Theorem 1.2.1. *Let $k \in \mathbb{Z}_{\geq 0}$ such that $(-1)^k\Delta < 0$ and let $F \in H^+_{-2k}$ with vanishing constant term $a^+(0)$.*

1. *The Millson theta lift $I_\Delta^M(F,\tau)$ is a harmonic Maass form in $H^+_{1/2-k}(\Gamma_0(4))$ satisfying the Kohnen plus space condition. Further, if F is weakly holomorphic, then so is $I_\Delta^M(F,\tau)$.*

2. *$I_\Delta^M(F,\tau)$ is related to the Shintani lift of $\xi_{-2k,z}F \in S_{2k+2}$ by*

$$\xi_{1/2-k,\tau} I_\Delta^M(F,\tau) = -4^k\sqrt{|\Delta|}\, I_\Delta^{Sh}(\xi_{-2k,z}F,\tau),$$

that is, the following diagram is commutative (up to scaling factors)

$$\begin{tikzcd} M^!_{-2k} \arrow[r] \arrow[d, "I_\Delta^M"] & H^+_{-2k} \arrow[r, "\xi_{-2k}"] \arrow[d, "I_\Delta^M"] & S_{2k+2} \arrow[d, "I_\Delta^{Sh}"] \\ M^!_{\frac{1}{2}-k} \arrow[r] & H^+_{\frac{1}{2}-k} \arrow[r, "\xi_{1/2-k}"] & S_{\frac{3}{2}+k}. \end{tikzcd}$$

3. *The Fourier expansion of $I_\Delta^M(F,\tau)$ is given by*

$$\begin{aligned} I_\Delta^M(F,\tau) = &\sum_{D<0} \frac{2}{\sqrt{|D|}} \left(\frac{1}{2\pi\sqrt{|\Delta D|}}\right)^k \mathrm{tr}^+_{R^k_{-2k}F,\Delta}(D) q^{-D} \\ &- \sum_{b>0} 2i\varepsilon \left(\frac{1}{2\pi i|\Delta|}\right)^k \sum_{n<0} \left(\frac{\Delta}{n}\right) a^+(nb)(4\pi n)^k q^{-|\Delta|b^2} \\ &- \sum_{D>0} \frac{1}{2(\pi D)^{k+1/2}|\Delta|^{k/2}} \overline{\mathrm{tr}_{\xi_{-2k}F,\Delta}(D)}\, \Gamma(1/2+k, 4\pi D v) q^{-D}, \end{aligned}$$

where $R_{-2k}^k = R_{-2} \circ R_{-4} \circ \cdots \circ R_{-2k}$ *is the iterated Maass raising operator and* ε *equals* 1 *or* i *according to whether* $\Delta > 0$ *or* $\Delta < 0$.

Remark 1.2.2.
1. The general results for harmonic Maass forms of higher level can be found in Proposition 3.4.1 and Theorem 3.4.3.

2. The assumption $a^+(0) = 0$ was imposed here to simplify the exposition in the introduction and will not be used in the main part of this work. If $a^+(0) \neq 0$ then $I_\Delta^M(F, \tau)$ also has a constant coefficient, and for $k = 0$ further non-holomorphic terms appear. In fact, for $k = 0$ the ξ-image of the Millson lift $I_\Delta^M(F, \tau)$ of a weakly holomorphic modular function F with non-vanishing constant coefficient turns out to be a linear combination of unary theta series of weight $3/2$. Thus, using the theta lift, one can obtain formulas for the coefficients of mock theta functions of weight $1/2$ as traces of modular functions, similarly as in [Alf14]. We apply this idea in Section 4.1.

3. In [BGK14], the authors studied a so-called Zagier lift, which (for level 1 and $k > 1$) maps weight $-2k$ to weight $1/2 - k$ harmonic Maass forms. The proof of the modularity of this lift uses the Fourier coefficients of non-holomorphic Poincaré series together with the fact that a harmonic Maass form of negative weight is uniquely determined by its principal part. Thus their proof does not work for $k = 0$. In fact, the Zagier lift agrees with our lift in level 1, so our theorem generalizes Proposition 6.2 of [BGK14] to arbitrary level and to $k = 0$, using a very different proof.

4. Integrating $\Theta_{M,k,\Delta}(\tau, z)$ in τ against a harmonic Maass form of weight $1/2-k$ yields a so-called locally harmonic Maass form of weight $-2k$. This lift was considered in [Höv12], [BKV13] and [Cra15], and it was shown that the resulting theta lift is related to the Shimura lift via the ξ-operator.

Example 1.2.3. Let $k = 0$ and $\Delta < 0$. For $m \in \mathbb{Z}_{\geq 0}$ we let $J_m \in M_0^!(\Gamma)$ denote the unique weakly holomorphic modular function for $\mathrm{SL}_2(\mathbb{Z})$ whose Fourier expansion starts $J_m = q^{-m} + O(q)$, e.g., $J_0 = 1$, $J_1 = j - 744$. The Δ-th twisted Millson lift of J_m is given by

$$I_\Delta^M(J_m, \tau) = 2\sum_{n \mid m} \left(\frac{\Delta}{m/n}\right) q^{\Delta n^2} + 2\sum_{D<0} \frac{1}{\sqrt{|D|}} \operatorname{tr}_{J_m,\Delta}^+(D) q^{-D} = 2\sum_{n \mid m} \left(\frac{\Delta}{m/n}\right) f_{|\Delta| n^2}.$$

where f_d (for a discriminant $-d < 0$) denotes the unique weakly holomorphic modular form of weight $1/2$ for $\Gamma_0(4)$ in the plus space with Fourier expansion $q^{-d} + O(q)$, compare [Zag02].

As a part of the proof of the above theorem, we show that the so-called Bruinier-Funke theta lift studied in [Alf15], which is a theta lift constructed from the $k = 0$ Millson theta function and suitable applications of iterated Maass raising and lowering operators, essentially agrees with the Millson theta lift on harmonic Maass forms of weight $-2k$. This identity of theta lifts is a bit surprising and interesting in its own right, but due to its quite technical appearence we chose not to state it in the introduction. We refer the reader to Theorem 3.2.3.

The relation between the Millson lift and the Shintani lift also yields an interesting criterion for the vanishing of the twisted L-function of a newform at the critical point.

Theorem 1.2.4. *Let $F \in H^+_{-2k}$, with vanishing constant term $a^+(0)$ if $k = 0$, such that $G = \xi_{-2k}F \in S_{2k+2}$ is a normalized newform. For $(-1)^k\Delta < 0$ the lift $I^M_\Delta(F, \tau)$ is weakly holomorphic if and only if $L(G, \chi_\Delta, k + 1) = 0$.*

Remark 1.2.5. For the general result regarding forms of higher level see Theorem 3.4.2. A version of this theorem for square-free level N and odd weight k has been proved in [Alf14], Theorem 1.1, using the same techniques. Further, the above theorem in the case of level 1 and $k > 0$ already appeared in [BGK14], Corollary 1.3.

The Fourier coefficients of the non-holomorphic part of the Millson theta lift in Theorem 1.2.1 involve cycle integrals of the cusp form $\xi_{-2k}F$, which reflects the relation between the Millson and the Shintani lift on the level of Fourier expansions. On the other hand, the fact that the Millson theta lift agrees up to some constant with a theta lift of the real-analytic modular function $R^k_{-2k}F$, see Theorem 3.2.3, suggests that the Fourier coefficients of the non-holomorphic part of the Millson lift should also be expressible in terms of cycle integrals of $R^k_{-2k}F$. Inspired by this idea, we prove the following identities between the cycle integrals of different types of modular forms.

Theorem 1.2.6. *Let $D > 0$ be a discriminant which is not a square and let $Q \in \mathcal{Q}_D$. Let $k \in \mathbb{Z}_{\geq 0}$ and $F \in H^+_{-2k}$. For $j \in \mathbb{Z}_{\geq 0}$ we have*

$$\mathcal{C}(R^{2j+1}_{-2k}F, Q) = \frac{1}{D^{k-j}} \frac{j!(k-j)!(2k)!}{k!(2k-2j)!} \overline{\mathcal{C}(\xi_{-2k}F, Q)},$$

where

$$\mathcal{C}(G, Q) = \int_{\Gamma_Q \backslash c_Q} G(z)Q(z, 1)^k dz$$

is the cycle integral of a function G transforming of weight $2k+2$, and $R^{2j+1}_{-2k} = R_{-2k+2j} \circ \cdots \circ R_{-2k}$ is the iterated Maass raising operator.

We prove this identity by a direct computation using Stokes' theorem and commutation relations for the differential operators involved.

Remark 1.2.7. 1. It is interesting to note that the cycle integral of $\xi_{-2k}F$ on the right-hand side does not depend on j. In particular, the cycle integrals of $R^{2j+1}_{-2k}F$ for different choices of j are related by a very simple explicit constant.

2. The general result is given in Corollary 3.6.2. By plugging in special values for j, e.g. $j = k$, we obtain further interesting formulas (see Corollaries 3.6.3 and 3.6.4), which were previously given in Theorem 1.1 from [BGK14] and Theorem 1.1 from [BGK15]. The above identity gives a unified proof for these two previously known, but seemingly unrelated results.

3. We also define a regularized cycle integral $\mathcal{C}^{\mathrm{reg}}(R_{-2k}^{2j+1}F, Q)$ in the case that the discriminant of Q is a square and the associated geodesic is infinite, and derive an analog of the above theorem in this situation, see Section 3.6.2.

1.3 Applications of the Millson and Kudla-Millson theta lifts

We discuss some new applications of the Kudla-Millson lift from [BF06] and the Millson lift studied in this work. This chapter is based on joint work with Jan Hendrik Bruinier, see [BS17].

Algebraic formulas for Ramanujan's mock theta functions

We give finite algebraic formulas for the coefficients of Ramanujan's order 3 mock theta functions $f(q)$ and $\omega(q)$ in terms of traces of CM values of a weakly holomorphic modular function (see Theorem 4.1.1). For example, we show that the coefficients $a_f(n), n \geq 1$, of Ramanujan's mock theta function

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} = 1 + \sum_{n=1}^{\infty} a_f(n) q^n \tag{1.3.1}$$

are given by

$$a_f(n) = -\frac{1}{\sqrt{24n-1}} \operatorname{Im}\left(\sum_{Q \in \mathcal{Q}_n} \frac{F(z_Q)}{\omega_Q} \right),$$

where

$$F(z) = -\frac{1}{40} \cdot \frac{E_4(z) + 4E_4(2z) - 9E_4(3z) - 36E_4(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} = q^{-1} - 4 - 83q + \ldots \tag{1.3.2}$$

is a $\Gamma_0(6)$-invariant weakly holomorphic modular function, $\mathcal{Q}_n$ is the (finite) set of $\Gamma_0(6)$-equivalence classes of positive definite integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ of discriminant $1 - 24n$ with $6 \mid a$ and $b \equiv 1(12)$, $z_Q \in \mathbb{H}$ is the CM point characterized by $Q(z_Q, 1) = 0$, and ω_Q is half the order of the stabilizer of Q in $\Gamma_0(6)$. Moreover, E_4 denotes the normalized Eisenstein series of weight 4 for Γ and $\eta = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function.

For the proof, we use Zwegers' [Zwe02] realization of Ramanujan's mock theta functions as the holomorphic parts of vector valued harmonic Maass forms of weight 1/2, then construct the corresponding harmonic Maass form as the Millson lift of F, and finally obtain the formula by comparing Fourier coefficients.

Rationality results for harmonic Maass forms

By applying the Kudla-Millson and the Millson theta lifts to a suitable weakly holomorphic input function, we construct harmonic Maass forms of weight 3/2 and 1/2 whose images under the differential operator $\xi_k = 2iv^k\overline{\frac{\partial}{\partial\overline{\tau}}}$ are vector valued unary theta functions of weight 1/2 and 3/2, and whose holomorphic parts (which are mock modular forms) are given by traces of CM values of the input function (see Theorem 4.2.4). This implies that these mock modular forms have rational coefficients (see Theorem 4.2.6), which in turn yields a rationality result for the holomorphic parts of harmonic Maass forms that map to the space of unary theta functions under ξ (see Theorem 4.2.8).

More specifically, we show that if f is a vector valued harmonic Maass form of weight 1/2 whose principal part is defined over a number field K, and whose shadow lies in the space of unary theta functions, then all coefficients of the holomorphic part of f lie in K. This contrasts a conjecture of Bruinier and Ono [BO10b], stating that if f is a harmonic Maass form of weight 1/2 whose shadow is orthogonal to the space of unary theta functions, then all but a set of density 0 of the non-vanishing coefficients of the holomorphic part of f should be transcendental.

Inner product formulas and Weyl vectors of Borcherds products

We use our ξ-preimages to evaluate the regularized Petersson inner product of a harmonic Maass form f and a unary theta function of weight 1/2 (see Theorem 4.3.1), and apply this to compute the Weyl vectors of the Borcherds product of f (see Corollary 4.3.4). For example, for $N = 1$ the Borcherds product associated to a weakly holomorphic modular form $f = \sum_{n\gg-\infty} c_f(n)q^n$ of weight 1/2 for $\Gamma_0(4)$ in the Kohnen plus space with rational coefficients and integral principal part is given by

$$\Psi(z,f) = q^{\rho_f}\prod_{n=1}^{\infty}(1-q^n)^{c_f(n^2)}, \qquad (q = e^{2\pi iz}),$$

where ρ_f is the so-called Weyl vector of f. The product converges for $\mathrm{Im}(z) \gg 0$ large enough and extends to a meromorphic modular form of weight $c_f(0)$ for Γ, whose divisor on $\mathbb{H}$ is a Heegner divisor. For higher level N, the orders of $\Psi(z,f)$ at the cusps of $\Gamma_0(N)$ are determined by Weyl vectors associated to the cusps. These vectors are essentially given by regularized inner products of f with a unary theta function of weight 1/2, and can be explicitly evaluated in terms of the coefficients of the holomorphic part of f and the coefficients of the holomorphic part of a ξ-preimage of the unary theta function. In particular, we show that all Weyl vectors associated to a harmonic Maass form with rational holomorphic coefficients are rational (see Corollary 4.3.3).

1.4 Borcherds lifts of harmonic Maass forms

In [Bor95], Borcherds defined a regularized theta lift which maps weakly holomorphic modular forms of weight $1/2$ to real analytic modular functions with logarithmic singularities at CM points. His results were generalized to twisted lifts of harmonic Maass forms which map to cusp forms under $\xi_{1/2}$ by Bruinier and Ono [BO10b]. We extend the lift to general harmonic Maass forms (which may map to weakly holomorphic modular forms under $\xi_{1/2}$) and give some applications. In the introduction, we restrict to modular forms for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ for simplicity, but in the body of the work we treat modular forms of arbitrary level $\Gamma_0(N)$.

A harmonic Maass form of weight $1/2$ for $\Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ which is annihilated by the invariant Laplace operator $\Delta_{1/2}$, transforms like a modular form of weight $1/2$ for $\Gamma_0(4)$, and grows at most linearly exponentially at the cusps of $\Gamma_0(4)$. Such a form can be written as a sum $f = f^+ + f^-$ with a holomorphic part f^+ and a non-holomorphic part f^- with Fourier expansions of the shape

$$\begin{aligned} f^+(\tau) &= \sum_{D\in\mathbb{Z}} c_f^+(D)e(D\tau) \\ f^-(\tau) &= c_f^-(0)\sqrt{v} + \sum_{D<0} c_f^-(D)\sqrt{v}\beta_{1/2}(4\pi|D|v)e(D\tau) \\ &\quad + \sum_{D>0} c_f^-(D)\sqrt{v}\beta_{1/2}^c(-4\pi Dv)e(D\tau), \end{aligned}$$

with coefficients $c_f^{\pm}(D) \in \mathbb{C}$, where $\beta_{1/2}(s) = \int_1^\infty e^{-st}t^{-1/2}dt$ and $\beta_{1/2}^c(s) = \int_0^1 e^{-st}t^{-1/2}dt$. Note that we use a slightly different normalization of the Fourier expansion compared to the one in (1.2.1), since it is more convenient here. We let $H_{1/2}$ denote the space of harmonic Maass forms satisfying the Kohnen plus space condition, which means that the Fourier expansion is supported on indices $D \equiv 0, 1(4)$.

Let Δ be a fundamental discriminant. For simplicity, we assume that $\Delta > 1$ in the introduction. We define the Borcherds lift $\Phi_\Delta(z, f)$ of a harmonic Maass form $f \in H_{1/2}$ by the regularized integral

$$\Phi_\Delta(f, z) = \mathrm{CT}_{s=0}\left[\lim_{T\to\infty}\int_{\mathcal{F}_T(4)}^{\mathrm{reg}} f(\tau)\overline{\Theta_\Delta(\tau, z)}v^{1/2-s}\frac{du\,dv}{v^2}\right],$$

where $\Theta_\Delta(\tau, z)$ is a twisted Siegel theta function which transforms in τ like a modular form of weight $1/2$ for $\Gamma_0(4)$ and is invariant in z under Γ, $\mathcal{F}_T(4)$ denotes a suitably truncated fundamental domain for $\Gamma_0(4)\backslash\mathbb{H}$, and $\mathrm{CT}_{s=0}\,F(s)$ denotes the constant term in the Laurent expansion at $s = 0$ of a function $F(s)$ which is meromorphic near $s = 0$. Borcherds [Bor98] proved that for $\Delta = 1$ and a weakly holomorphic modular form $f \in M^!_{1/2}$ the regularized theta lift $\Phi_\Delta(f, z)$ defines a Γ-invariant real analytic function with logarithmic singularities at certain CM points in $\mathbb{H}$, which are determined by the principal part of f. Bruinier and Ono [BO10b] showed that this result remains true for twisted Borcherds lifts of harmonic Maass forms $f \in H^+_{1/2}$, i.e. forms which map to cusp forms under $\xi_{1/2}$. One of the main aims of the present work is to generalize the Borcherds lift $\Phi_\Delta(z, f)$ to the full space $H_{1/2}$.

We let $H^+_\Delta(f)$ be the set of all CM points z_Q corresponding to quadratic forms $Q \in \mathcal{Q}_{\Delta D}$ with $D < 0$ such that $c^+_f(D) \neq 0$, and we let $H^-_\Delta(f)$ be the union of all geodesics c_Q corresponding to quadratic forms $Q \in \mathcal{Q}_{\Delta D}$ with $D > 0$ such that $c^-_f(D) \neq 0$. We obtain the following extension of the Borcherds lift on the full space $H_{1/2}$.

Theorem 1.4.1. *Let $\Delta > 1$ be a fundamental discriminant. For $f \in H_{1/2}$ the Borcherds lift $\Phi_\Delta(f, z)$ defines a Γ-invariant harmonic function on $\mathbb{H} \setminus (H^+_\Delta(f) \cup H^-_\Delta(f))$. It has 'logarithmic singularities' at the CM points in $H^+_\Delta(f)$, and 'arcsin singularities' along the geodesics in $H^-_\Delta(f)$. More precisely, this means that for $z_0 \in H^+_\Delta(f) \cup H^-_\Delta(f)$ the function*

$$\Phi_\Delta(f, z) - \sum_{D<0} c^+_f(D) \sum_{\substack{Q \in \mathcal{Q}_{\Delta D} \\ z_0 = z_Q}} \chi_\Delta(Q) \log |az^2 + bz + c|$$
$$+ \sum_{D>0} \frac{c^-_f(D)}{\sqrt{D}} \sum_{\substack{Q \in \mathcal{Q}_{\Delta D} \\ z_0 \in c_Q}} \chi_\Delta(Q) \arcsin\left(\frac{1}{\sqrt{1 + \frac{1}{\Delta D y^2}(a|z|^2 + bx + c)^2}} \right).$$

can be continued to a real analytic function near z_0. Here χ_Δ is the genus character defined in (1.2.2). Note that all the above sums are finite.

We refer the reader to Theorem 5.1.1 for the general result.

Remark 1.4.2. The logarithmic singularities imply that the Borcherds lift blows up at the Heegner points $z_Q \in H^+_\Delta(f)$, and the arcsin singularities show that it is continuous but not differentiable at points on the geodesics $c_Q \subset H^-_\Delta(f)$.

Using non-holomorphic Maass-Poincaré series one can always write a harmonic Maass form $f \in H_{1/2}$ as $f = f_1 + f_2$ where $f_1, f_2 \in H_{1/2}$ satisfy $c^+_{f_1}(n) = 0$ for all $n < 0$ and $c^-_{f_2}(n) = 0$ for all $n \geq 0$. In particular, we have $f_2 \in H^+_{1/2}$, and since the Borcherds lift of harmonic Maass forms in $H^+_{1/2}$ has already been investigated by Bruinier and Ono [BO10b], we assume from now on that $c^+_f(n) = 0$ for all $n < 0$. In this case, the Borcherds lift $\Phi_\Delta(f, z)$ only has singularities along the geodesics in $H^-_\Delta(f)$. Furthermore, the Fourier expansion of $\Phi_\Delta(f, z)$ can be stated as follows.

Proposition 1.4.3. *Let $\Delta > 1$ be a fundamental discriminant and let $f \in H_{1/2}$ such that $c^+_f(n) = 0$ for all $n < 0$. Then for $z \in \mathbb{H} \setminus H^-_\Delta(f)$ the Borcherds lift of f has the*

Fourier expansion

$$\begin{aligned}\Phi_\Delta(f,z) = &-4\sum_{m=1}^{\infty} c_f^+(\Delta m^2)\sum_{b(\Delta)}\left(\frac{\Delta}{b}\right)\log|1-e(mz+b/\Delta)| \\ &+ \sqrt{\Delta}\,L(1,\chi_\Delta)\left(2c_f^+(0)+y\,c_f^-(0)\right) \\ &- 4\sum_{D>0}\frac{c_f^-(D)}{\sqrt{D}}\sum_{\substack{Q\in\mathcal{Q}_{\Delta D}\\ a>0}}\chi_\Delta(Q)\mathbf{1}_Q(z)\left(\arctan\left(\frac{y\sqrt{\Delta D}}{a|z|^2+bx+c}\right)+\frac{\pi}{2}\right),\end{aligned}$$

where $\mathbf{1}_Q(z)$ denotes the characteristic function of the bounded component of $\mathbb{H}\setminus c_Q$.

For the general result, see Proposition 5.2.2.

Remark 1.4.4.
1. For $Q\in\mathcal{Q}_{\Delta D}$ with $a>0$ the corresponding geodesic c_Q is a semi-circle centered at the real line which divides $\mathbb{H}$ into a bounded and an unbounded connected component, so the characteristic function $\mathbf{1}_Q$ makes sense.
2. The sum over D in the third line is finite since f has a finite principal part. The sum over $Q\in\mathcal{Q}_{\Delta D}$ is locally finite since each point $z\in\mathbb{H}$ lies in the bounded component of $\mathbb{H}\setminus c_Q$ for finitely many geodesics $c_Q\in\mathcal{Q}_{\Delta D}$, and it vanishes for $y\gg 0$ large enough since the imaginary parts of points lying on geodesics c_Q for $Q\in\mathcal{Q}_{\Delta D}$ are smaller than $\sqrt{\Delta D}$.
3. We have $z\in c_Q$ for $Q=[a,b,c]$ if and only if $a|z|^2+bx+c=0$. Further, for $a>0$ a point $z\in\mathbb{H}$ lies in the inside of the bounded component of $\mathbb{H}\setminus c_Q$ if and only if $a|z|^2+bx+c<0$. Since $\lim_{x\to-\infty}\arctan(x)=-\frac{\pi}{2}$, we see from the Fourier expansion that $\Phi_\Delta(f,z)$ is continuous. However, computing the derivative of the above expansion for $z\in\mathbb{H}\setminus H_\Delta^-(f)$ shows that the third line is not differentiable at points $z\in H_\Delta^-(f)$. More precisely, the derivative of $\Phi_\Delta(f,z)$ has jumps along the geodesics in $H_\Delta^-(f)$.

We apply the (derivative of the) Borcherds lift to certain interesting harmonic Maass forms of weight $1/2$ for $\Gamma_0(4)$, in order to construct modular integrals of weight 2 with rational period functions. In [DIT11], Duke, Imamoglu and Tóth constructed a basis $\{h_d\}$ (indexed by discriminants $d>0$) of $H_{1/2}$, which under $\xi_{1/2}$ maps to a basis $\{g_d\}$ of the space of weakly holomorphic modular forms of weight $3/2$ for $\Gamma_0(4)$. More precisely, the g_d are the generating series of traces of singular moduli, see [Zag02]. The coefficients of the h_d are given by traces of CM values and traces of (regularized) cycle integrals of weakly holomorphic modular functions for Γ. For example, the Fourier expansion of the function $h=h_1$ is given by

$$h(\tau)=\frac{1}{2\pi}\sum_{D>0}\mathrm{tr}_J(D)q^D+2\sqrt{v}\beta_{1/2}^c(-4\pi v)q-8\sqrt{v}+2\sqrt{v}\sum_{D<0}\mathrm{tr}_J^+(D)\beta_{1/2}(4\pi|D|v)q^D.$$

Here the traces for $D > 0$ being a square need to be regularized as explained in [BFI15]. The harmonic Maass form h does in general not map to a cusp form but to a weakly holomorphic modular form under $\xi_{1/2}$, so it is interesting to apply our extension of the Borcherds lift to it. The coefficients $c_h^+(D)$ for $D \leq 0$ vanish, so the Borcherds lift $\Phi_\Delta(h,z)$ is a harmonic Γ-invariant function on $\mathbb{H} \setminus H_\Delta^-(h)$ with arcsin singularities along the geodesics in $H_\Delta^-(h)$. In this case, the latter set is just the union of all geodesics c_Q for $Q \in \mathcal{Q}_\Delta$. Hence the derivative $\Phi'_\Delta(h,z) = \frac{\partial}{\partial z}\Phi_\Delta(h,z)$ is a holomorphic function on $\mathbb{H} \setminus H_\Delta^-(h)$ transforming like a modular form of weight 2 for Γ. Moreover, it turns out that $\Phi'_\Delta(h,z)$ has jump singularities along the geodesics in $H_\Delta^-(h)$, and admits a nice Fourier expansion.

Proposition 1.4.5. *Let $\Delta > 1$ be a fundamental discriminant. The derivative $\Phi'_\Delta(h,z)$ of the Borcherds lift of h is a holomorphic function on $\mathbb{H} \setminus H_\Delta^-(h)$ which transforms like a modular form of weight 2 for Γ. For $z \in \mathbb{H} \setminus H_\Delta^-(h)$ it has the expansion*

$$\begin{aligned}&\frac{1}{4\pi i\sqrt{\Delta}}\Phi'_\Delta(h,z)\\ &= \frac{1}{2\pi}\operatorname{tr}_1(\Delta) + \frac{1}{2\pi}\sum_{n=1}^\infty \left(\sum_{m\mid n}\left(\frac{\Delta}{n/m}\right) m \operatorname{tr}_J(\Delta m^2)\right) e(nz) + \frac{1}{\pi}\sum_{\substack{Q\in\mathcal{Q}_\Delta\\ a>0}} \frac{\mathbf{1}_Q(z)}{Q(z,1)},\end{aligned}$$

where $\mathbf{1}_Q$ denotes the characteristic function of the bounded component of $\mathbb{H} \setminus c_Q$.

The result for general harmonic Maass forms $f \in H_{1/2}$ of higher level is given in Proposition 5.3.3 and Corollary 5.3.5.

Remark 1.4.6. The Fourier series over n is holomorphic on $\mathbb{H}$, whereas the sum over Q has jump singularities along the geodesics c_Q with $Q \in \mathcal{Q}_\Delta$. Again, the sum over Q is locally finite and vanishes for $y \gg 0$ large enough.

In [DIT11], Theorem 5, the authors proved that the generating series

$$F_\Delta(z) = \frac{1}{\pi}\sum_{m=0}^\infty \operatorname{tr}_{J_m}(\Delta) e(mz),$$

with $J_m(z) = q^{-m} + O(q) \in M_0^!$, e.g. $J_0 = 1$ and $J_1 = J$, defines a holomorphic function on $\mathbb{H}$ which transforms as

$$z^{-2}F_\Delta\left(-\frac{1}{z}\right) - F_\Delta(z) = \frac{2}{\pi}\sum_{\substack{Q\in\mathcal{Q}_\Delta\\ c<0<a}} \frac{1}{Q(z,1)}, \tag{1.4.1}$$

so $F_\Delta(z)$ is a holomorphic modular integral of weight 2 with holomorphic rational period functions in the sense of [Kno74], whose definition we now recall. A 1-cocycle for Γ with values in the set R of rational functions on $\mathbb{C}$ is a map $q : \Gamma \to R, M \mapsto q_M$ satisfying

$$q_{MM'} = q_M|_2 M' + q_{M'}$$

for all $M, M' \in \Gamma$, and a 1-coboundary is a 1-cocycle which can be written as $q_M = r|_2 M - r$ for some fixed rational function r. We call q parabolic it $q_T = 0$, and denote the corresponding parabolic cohomology group by $H^1_{par}(\Gamma, R)$. For example, equation (1.4.1) easily implies that the map

$$q_M^\Delta = \sum_{\substack{Q \in \mathcal{Q}_\Delta \\ a_{QM^{-1}} < 0 < a_Q}} \frac{1}{Q(z,1)}$$

defines a parabolic 1-cocycle for Γ. Refining this construction, one can show that for any Γ-class $\mathcal{A}$ of primitive indefinite binary quadratic forms the map

$$q_M^{\mathcal{A}} = \sum_{\substack{Q \in \mathcal{A} \\ a_{QM^{-1}} < 0 < a_Q}} \frac{1}{Q(z,1)}$$

defines a parabolic 1-cocycle (see also [DIT17]). Choie and Zagier [CZ93] constructed an explicit basis $\{r^{\mathcal{A}}\}$ for $H^1_{par}(\Gamma, R)$, which is labelled by the Γ-classes of primitive indefinite binary quadratic forms, and which has the property that $q^{\mathcal{A}} = r^{\mathcal{A}} + r^{-\mathcal{A}}$. More generally, the structure of $H^1_{par}(G, R)$ for any finite index subgroup G of Γ has been determined using cohomological methods by Ash [Ash89]. It would be desirable to construct an explicit basis of $H^1_{par}(G, R)$ for all finite index subgroups G of Γ, and we hope to come back to this problem in the future.

A modular integral (of weight 2 with rational period functions) for a 1-cocycle q is a holomorphic function $F = \sum_{n \geq 0} a(n) q^n : \mathbb{H} \to \mathbb{C}$ such that

$$q_M = F|_2 M - F$$

for each $M \in \Gamma$. For example, F_Δ is a modular integral for (a multiple of) q^Δ. The existence of modular integrals was proven by Knopp [Kno74], and the connection to generating series of traces of cycle integrals of weakly holomorphic modular forms was discovered by Duke, Imamoglu and Tóth [DIT10], [DIT11], [DIT17].

Returning to the derivative of the Borcherds lift of h, we note that

$$\operatorname{tr}_{J_m}(\Delta) = \sum_{d|m} \left(\frac{\Delta}{m/d}\right) d \operatorname{tr}_{J_1}(\Delta d^2), \tag{1.4.2}$$

compare [Zag81], pp. 290–292, so $F_\Delta(z)$ in fact agrees with $\Phi'_\Delta(h, z)$ up to some constant factor if $y \gg 0$ is sufficiently large. The transformation behaviour of the singular part in the Fourier expansion of $\Phi'_\Delta(h,z)$ can easily be determined, so we can recover (1.4.1) from Proposition 1.4.5. Further, using the Borcherds lift we generalize the construction of modular integrals of weight 2 with rational period functions from [DIT11] to higher level, see Proposition 5.4.1. The coefficients of our modular integrals are linear combinations of Fourier coefficients of the holomorphic parts of harmonic Maass forms f of weight 1/2. Choosing f as the image of a theta lift of a harmonic Maass form F of weight 0 studied by Bruinier, Funke and Imamoglu [BFI15], we obtain modular integrals whose coefficients are linear combinations of traces of cycle integrals of F. In fact, the construction of F_Δ as a theta lift and its generalizations to higher level were our main motivation to extend the Borcherds lift to the full space $H_{1/2}$.

Bruinier and Ono [BO10b] defined a twisted Borcherds product associated to a harmonic Maass form $f \in H^+_{1/2}$ with real coefficients $c_f^+(D)$ for all D, and $c_f^+(D) \in \mathbb{Z}$ for $D \leq 0$. For $\Delta > 1$ a fundamental discriminant and $y \gg 0$ sufficiently large the twisted Borcherds lift of f is given by

$$\Psi_\Delta(f,z) = \prod_{m=1}^{\infty} \prod_{b(\Delta)} [1 - e(mz + b/\Delta)]^{\left(\frac{\Delta}{b}\right) c_f^+(\Delta m^2)}.$$

It has a meromorphic continuation to $\mathbb{H}$ with roots and poles at CM points corresponding to the principal part of f, and it transforms like a modular form of weight 0 with some unitary character for Γ. We will define Borcherds products associated to general harmonic Maass forms $f \in H_{1/2}$. For simplicity, in the introduction we only consider the harmonic Maass form πh. The general result is given in Theorem 5.4.10.

Theorem 1.4.7. *Let $\Delta > 1$ be a fundamental discriminant. Then the infinite product*

$$\Psi_\Delta(z) = e\left(-\sqrt{\Delta}\,\mathrm{tr}_1(\Delta) z\right) \prod_{m=1}^{\infty} \prod_{b(\Delta)} [1 - e(mz + b/\Delta)]^{\left(\frac{\Delta}{b}\right) \mathrm{tr}_J(\Delta m^2)}$$

converges to a holomorphic function on $\mathbb{H}$. Its logarithmic derivative is given by

$$\frac{\partial}{\partial z} \log(\Psi_\Delta(z)) = -2\pi^2 i \sqrt{\Delta} F_\Delta(z).$$

Further, it transforms as

$$\begin{aligned}
\Psi_\Delta(z+1) &= e\left(-\sqrt{\Delta}\,\mathrm{tr}_1(\Delta)\right) \Psi_\Delta(z), \\
\Psi_\Delta\left(-\frac{1}{z}\right) &= e\left(-2 \sum_{\substack{Q \in \mathcal{Q}_\Delta \\ c<0<a}} \left(\log\left(\frac{z - w_Q}{i - w_Q}\right) - \log\left(\frac{z - w'_Q}{i - w'_Q}\right)\right)\right) \Psi_\Delta(z),
\end{aligned}$$

where $w_Q > w'_Q$ denote the real endpoints of the geodesic c_Q.

2 Preliminaries

2.1 Quadratic forms, lattices and the Weil representation

We start with the basic facts about quadratic modules, lattices and the Weil representation, and we give a brief introduction to the construction of Siegel theta functions associated to an indefinite even lattice and a homogeneous polynomial. There are many good books about the theory of quadratic forms, for example [CS99] and [Ebe02]. The exposition on the Weil representation follows [Bru02], and the section on theta functions is based on [Bor98].

2.1.1 Quadratic modules

In the following, we let R be a principal ideal domain and

$$M = Rb_1 \oplus \cdots \oplus Rb_m$$

with $b_1, \ldots, b_m \in M$ a finitely generated free R-module. A map $Q : M \to R$ is called a quadratic form on M if it satisfies

$$Q(rx) = r^2 Q(x)$$

for all $r \in R$ and $x \in M$, and if the map

$$(x, y) = Q(x + y) - Q(x) - Q(y)$$

is a symmetric bilinear form. If Q is non-degenerate, i.e., if $(x, y) = 0$ for all $y \in M$ implies $x = 0$, then the pair (M, Q) is called a quadratic module, or a quadratic space if R is a field. The quadratic form Q is called positive (negative) definite if $Q(x) > 0$ ($Q(x) < 0$) holds for all $x \in M \setminus \{0\}$. The Gram matrix of (M, Q) with respect to a basis $B = (b_1, \ldots, b_m)$ is defined as the matrix whose (i, j)-th entry is (b_i, b_j).

An isometry between two quadratic modules (M, Q) and (M', Q') is an injective R-linear map $\sigma : M \to M'$ with

$$Q'(\sigma(x)) = Q(x)$$

for all $x \in M$. The set of all isometries from M to itself is called the orthogonal group of M and is denoted by $O(M)$.

For a pair (p, q) of non-negative integers we let $\mathbb{R}^{p,q}$ denote the real quadratic space $\mathbb{R}^{p+q}$ with the quadratic form

$$(x_1, \ldots, x_{p+q}) \mapsto x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2.$$

We let $O(p, q)$ be its orthogonal group. Silvester's law of inertia asserts that every real quadratic space (V, Q) is isometrically isomorphic to $\mathbb{R}^{p,q}$ for a unique tuple (p, q), called the signature of (V, Q). If (V, Q) is a rational quadratic space, we define its signature as the signature of the real quadratic space $V(\mathbb{R}) = V \otimes \mathbb{R}$ with $Q(x \otimes r) = r^2 Q(x)$.

2.1.2 The Weil representation associated to an even lattice

In this subsection we let (V, Q) be a rational quadratic space of dimension m. A lattice L in V is a $\mathbb{Z}$-module of rank m, i.e., a subset of the form

$$L = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_m$$

for a basis $(b_1, \ldots, b_m)$ of V. The signature of L is defined as the signature of $V(\mathbb{R})$. The determinant $\det(L)$ of L is the determinant of any Gram matrix of L. It is independent of the choice of basis of L. A lattice L is called integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in L$, and it is called even if $(x, x) \in 2\mathbb{Z}$ for all $x \in L$. Note that every even lattice is integral. An element $x \in L$ is called primitive if $\mathbb{Q}x \cap L = \mathbb{Z}x$. It is called isotropic if $Q(x) = 0$. The dual lattice of L is defined by

$$L' = \{x \in V : (x, y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

If $L = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_m$, then $L' = \mathbb{Z}b_1' \oplus \cdots \oplus \mathbb{Z}b_m'$, where $(b_1', \ldots, b_m')$ denotes the basis of V defined by $(b_i, b_j') = \delta_{i,j}$. Thus L' is indeed a lattice. The level of L is defined as the smallest positive integer N such that $NQ(x) \in \mathbb{Z}$ for all $x \in L'$.

Let L be an even lattice of level N. Then $L \subseteq L'$ and $NL' \subseteq L$, and by the elementary divisors theorem the quotient L'/L is a finite abelian group of order $|\det(L)|$, called the discriminant group of L. The quadratic form Q on L induces a well-defined map

$$Q : L'/L \to \mathbb{Q}/\mathbb{Z}, \quad Q(x + L) = Q(x) \bmod \mathbb{Z}.$$

It satisfies $Q(nx) = n^2 Q(x) \bmod \mathbb{Z}$ for $n \in \mathbb{Z}$ and $x \in L'/L$, and the map

$$(x + L, y + L) = Q(x + y) - Q(x) - Q(y) \bmod \mathbb{Z}$$

defines a non-degenerate symmetric $\mathbb{Z}$-bilinear form on L'/L. The level of the discriminant group L'/L is defined as the level of L, i.e., it is the smallest positive integer N such that $NQ(h) = 0 \bmod \mathbb{Z}$ for all $h \in L'/L$.

For each $h \in L'/L$ we introduce a symbol $\mathfrak{e}_h$, and we let

$$\mathbb{C}[L'/L] = \left\{ \sum_{h \in L'/L} \lambda_h \mathfrak{e}_h : \lambda_h \in \mathbb{C} \right\}$$

be the group ring of L'/L. The multiplication on $\mathbb{C}[L'/L]$ is defined by $\mathfrak{e}_h \cdot \mathfrak{e}_{h'} = \mathfrak{e}_{h+h'}$. The natural inner product on $\mathbb{C}[L'/L]$ is given by

$$\left\langle \sum_{h \in L'/L} \lambda_h \mathfrak{e}_h, \sum_{h \in L'/L} \mu_h \mathfrak{e}_h \right\rangle = \sum_{h \in L'/L} \lambda_h \overline{\mu}_h.$$

We let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{R})$, realized as the set of pairs (M, ϕ), where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that

$$\phi(\tau)^2 = c\tau + d.$$

The group structure on $\mathrm{Mp}_2(\mathbb{R})$ is defined by

$$(M, \phi(\tau)) \cdot (M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We let $\widetilde{\Gamma} = \mathrm{Mp}_2(\mathbb{Z})$ denote the preimage of $\mathrm{SL}_2(\mathbb{Z})$ under the natural covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is generated by the elements

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \quad \text{and} \quad T = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Further, we let $\widetilde{\Gamma}_\infty$ be the subgroup of $\widetilde{\Gamma}$ generated by T.

Let $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. The Weil representation ρ_L of $\widetilde{\Gamma}$ associated to the lattice L is defined on the generators S and T of $\widetilde{\Gamma}$ by

$$\begin{aligned} \rho_L(T)\mathfrak{e}_h &= e(Q(h))\mathfrak{e}_h, \\ \rho_L(S)\mathfrak{e}_h &= \frac{e((q-p)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h', h))\mathfrak{e}_{h'}. \end{aligned}$$

We let ρ_L^* denote the dual Weil representation, which acts by the complex conjugate of the formula for the action of ρ_L defined above. The Weil representation factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $p - q$ is even, and a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $p - q$ is odd. Further, it is unitary with respect to the inner product on $\mathbb{C}[L'/L]$, i.e., it satisfies

$$\langle \rho_L((M,\phi))\mathfrak{e}_h, \rho_L((M,\phi))\mathfrak{e}_{h'}\rangle = \langle \mathfrak{e}_h, \mathfrak{e}_{h'}\rangle$$

for all $(M,\phi) \in \widetilde{\Gamma}$ and $h, h' \in L'/L$. The element

$$Z = S^2 = (ST)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right)$$

acts as

$$\rho_L(Z)\mathfrak{e}_h = i^{q-p}\mathfrak{e}_{-h}.$$

Explicit formulas for the actions of various congruence subgroups in the Weil representation can be found in [Sch09] for even signature and in [Str13] for odd signature.

2.1.3 Theta functions associated to indefinite lattices

Let (V,Q) be a rational quadratic space of signature (p,q). We let D be the Grassmannian of p-dimensional subspaces of $V(\mathbb{R}) = V \otimes \mathbb{R}$ on which Q is positive definite, i.e.,

$$D = \{z \subseteq V(\mathbb{R}) : \dim(z) = p, Q|_z > 0\}.$$

By Witt's theorem the orthogonal group $O(V(\mathbb{R})) \cong O(p,q)$ acts transitively on D, so if we pick a base point $z_0 \in D$ and let $O(V(\mathbb{R}))_{z_0} \cong O(p) \times O(q)$ denote its stabilizer, we have a bijection

$$D \cong O(V(\mathbb{R}))/O(V(\mathbb{R}))_{z_0} \cong O(p,q)/(O(p) \times O(q)),$$

which endows D with the structure of a smooth manifold. It admits a complex structure if and only if the signature of V is $(2,q)$ or $(p,2)$. For an explicit example we refer to Section 2.2 where an identification of the Grassmannian of positive definite lines in a quadratic space of signature $(1,2)$ with the complex upper half-plane $\mathbb{H}$ is given.

Let $L \subseteq V$ be an even lattice. The Siegel theta function associated to the lattice L is the $\mathbb{C}[L'/L]$-valued function on $\mathbb{H} \times D$ defined by

$$\Theta(\tau, z) = \mathrm{Im}(\tau)^{q/2} \sum_{h \in L'/L} \sum_{X \in h+L} e(\tau Q(X_z) + \bar{\tau} Q(X_{z^\perp}))\mathfrak{e}_h,$$

where X_z and $X_{z^\perp}$ denote the orthogonal projections of X to z and $z^\perp$, respectively. The theta function is a smooth function in τ and z which is invariant in z under any subgroup of $O(L)$ fixing the classes of L'/L, and it satisfies the transformation formula

$$\Theta(M\tau, z) = \phi(\tau)^{p-q}\rho_L(M,\phi)\Theta(\tau, z)$$

for all $(M,\phi)\in\widetilde{\Gamma}$, which can be proven using Poisson summation. In fact, the transformation formula for $\Theta(\tau,z)$ naturally leads to the definition of the Weil representation ρ_L. If L is positive definite, then the Grassmannian D consists of a single point, and the Siegel theta function reduces to the usual theta function associated to L.

Borcherds [Bor98] defined a more general theta function by associating to a (possibly complex valued) polynomial p on $\mathbb{R}^{p,q}$ and an isometry $v: V(\mathbb{R})\to\mathbb{R}^{p,q}$ the function

$$\Theta(\tau,v,p)=\sum_{h\in L'/L}\sum_{X\in h+L}\left(\exp\left(-\frac{\Delta}{8\pi\operatorname{Im}(\tau)}\right)p\right)(v(X))\cdot e(\tau Q(X_{v^+})+\bar{\tau}Q(X_{v^-}))\mathfrak{e}_h,$$

where

$$\exp\left(-\frac{\Delta}{8\pi\operatorname{Im}(\tau)}\right)p=\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{\Delta^k p}{(-8\pi\operatorname{Im}(\tau))^k},\qquad \Delta=\sum_{i=1}^{p+q}\frac{\partial^2}{\partial x_i^2},$$

is again a polynomial on $\mathbb{R}^{p,q}$, v^+ denotes the preimage under v of $\mathbb{R}^p\cong\{x\in\mathbb{R}^{p,q}: x_{p+1}=\cdots=x_{p+q}=0\}\subseteq\mathbb{R}^{p,q}$, and $v^-=(v^+)^\perp$. Note that v^+ defines an element in the Grassmannian D, and for $p=1$ we obtain

$$\operatorname{Im}(\tau)^{q/2}\Theta(\tau,v,1)=\Theta(\tau,v^+),$$

but for an arbitrary polynomial p the function $\Theta(\tau,v,p)$ will in general not define a function on D.

We call p harmonic if $\Delta p=0$, and homogeneous of degree (m^+,m^-) if it is homogeneous of degree m^+ in the first p variables and of degree m^- in the last q variables of $\mathbb{R}^{p,q}$. The following result can again be proven using Poisson summation.

Theorem 2.1.1 ([Bor98], Theorem 4.1)**.** *If p is a polynomial on $\mathbb{R}^{p,q}$ which is homogeneous of degree (m^+,m^-) then*

$$\Theta(M\tau,v,p)=\phi(\tau)^{p+2m^+}\overline{\phi(\tau)}^{q+2m^-}\rho_L(M,\phi)\Theta(\tau,v,p)$$

for $(M,\phi)\in\widetilde{\Gamma}$.

2.2 A quadratic space of signature $(1,2)$

We now fix the setup which will be used throughout this thesis. The exposition follows [BF06] and [Alf15].

2.2.1 The Grassmannian model of the upper half-plane

For a positive integer N we consider the rational quadratic space

$$V = \left\{ X = \begin{pmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{pmatrix}; x_1, x_2, x_3 \in \mathbb{Q} \right\}$$

with the quadratic form

$$Q(X) = N\det(X) = N(-x_2^2 - x_1x_3)$$

and the associated bilinear form

$$(X, Y) = -N\mathrm{tr}(XY) = -N(2x_2y_2 + x_1y_3 + x_3y_3).$$

The vectors

$$e_1 = \frac{1}{\sqrt{2N}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_2 = \frac{1}{\sqrt{2N}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \frac{1}{\sqrt{2N}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form an orthogonal basis of $V(\mathbb{R})$ with

$$(e_1, e_1) = 1, \quad (e_2, e_2) = (e_3, e_3) = -1,$$

so V has signature $(1, 2)$. We endow V with the orientation induced by this basis, i.e., a basis of V will be called positively oriented if the change of basis matrix to the basis (e_1, e_2, e_3) has positive determinant.

We identify the Grassmannian D of positive lines in $V(\mathbb{R})$ with the complex upper half-plane $\mathbb{H}$ by associating to $z = x + iy \in \mathbb{H}$ the positive line generated by

$$X_1(z) = \frac{1}{\sqrt{2N}y} \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}.$$

Note that $(X_1(z), X_1(z)) = 1$. The group $\mathrm{SL}_2(\mathbb{R})$ acts as isometries on $V(\mathbb{R})$ by

$$gX = gXg^{-1},$$

and it acts on $\mathbb{H}$ by fractional linear transformations. The identification above is $\mathrm{SL}_2(\mathbb{R})$-equivariant, that is, $gX_1(z) = X_1(gz)$ for $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$.

2.2.2 Cusps and truncated modular surfaces

Let L be an even lattice in the rational quadratic space V of signature $(1, 2)$ defined above, and let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that takes L to itself and acts trivially on the discriminant group L'/L. We identify the set of isotropic lines $\mathrm{Iso}(V)$ in

V with $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ via

$$\psi : P^1(\mathbb{Q}) \to \mathrm{Iso}(V), \quad \psi((\alpha : \beta)) = \mathrm{span}\left(\begin{pmatrix} \alpha\beta & \alpha^2 \\ -\beta^2 & -\alpha\beta \end{pmatrix}\right).$$

The map ψ is a bijection and satisfies $g\psi((\alpha : \beta)) = \psi(g(\alpha : \beta))$ for $g \in \mathrm{SL}_2(\mathbb{Q})$. Thus, the cusps of M, i.e., the Γ-classes of $P^1(\mathbb{Q})$, can be identified with the Γ-classes of $\mathrm{Iso}(V)$.

If we set $\ell_\infty := \psi(\infty)$, then ℓ_∞ is spanned by $X_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. For $\ell \in \mathrm{Iso}(V)$ we pick $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_\ell \ell_\infty = \ell$. An element of ℓ will be called positively oriented if it is a positive multiple of $\sigma_\ell X_0$. We let $\overline{\Gamma}_\ell$ be the stabilizer of ℓ in $\overline{\Gamma} = \Gamma/\{\pm 1\}$. Then

$$\sigma_\ell^{-1}\overline{\Gamma}_\ell \sigma_\ell = \left\{\begin{pmatrix} 1 & n\alpha_\ell \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}$$

for some $\alpha_\ell \in \mathbb{Q}_{>0}$ which we call the width of the cusp ℓ. For each ℓ, there is a $\beta_\ell \in \mathbb{Q}_{>0}$ such that $\left(\begin{smallmatrix} 0 & \beta_\ell \\ 0 & 0 \end{smallmatrix}\right)$ is a primitive element of $\ell_\infty \cap \sigma_\ell^{-1} L$. We write $\varepsilon_\ell = \alpha_\ell/\beta_\ell$. The quantities α_ℓ, β_ℓ and ε_ℓ only depend on the Γ-class of ℓ.

We let

$$M = \Gamma \backslash D \cong \Gamma \backslash \mathbb{H}$$

be the modular curve for Γ. We compactify M to a compact Riemann surface $\overline{M}$ by adding a point for each cusp $\ell \in \Gamma \backslash \mathrm{Iso}(V)$, and we denote this point again by ℓ. Write $q_\ell = \exp(2\pi i \sigma_\ell^{-1} z/\alpha_\ell)$ for the chart around ℓ. We define $D_{1/T} = \{w \in \mathbb{C} : |w| < \frac{1}{2\pi T}\}$ for $T > 0$. Note that if T is sufficiently big, then the inverse images $q_\ell^{-1} D_{1/T}$ are disjoint in M. We define the truncated modular curve by

$$M_T = \overline{M} \backslash \coprod_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} q_\ell^{-1} D_{1/T}. \tag{2.2.1}$$

A fundamental domain for M_T can be constructed as follows. We let

$$\mathcal{F} = \{z \in \mathbb{H} : |x| \leq 1/2, |z| \geq 1\}$$

be the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ and we let

$$\mathcal{F}_T = \{z \in \mathbb{H} : |x| \leq 1/2, |z| \geq 1, y \leq T\}$$

be a truncated fundamental domain. For $a \in \mathbb{N}$ we write

$$\mathcal{F}^a = \bigcup_{j=0}^{a-1} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \mathcal{F}, \qquad \mathcal{F}_T^a = \bigcup_{j=0}^{a-1} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \mathcal{F}_T.$$

A fundamental domain for $M = \Gamma\backslash\mathbb{H}$ is given by

$$\mathcal{F}(\Gamma) = \bigcup_{\ell \in \Gamma\backslash \operatorname{Iso}(V)} \sigma_\ell \mathcal{F}^{\alpha_\ell},$$

and a fundamental domain for M_T is given by

$$\mathcal{F}(\Gamma)_T = \bigcup_{\ell \in \Gamma\backslash \operatorname{Iso}(V)} \sigma_\ell \mathcal{F}_T^{\alpha_\ell}.$$

2.2.3 Heegner points and geodesics

For $X \in V$ with $Q(X) = m \in \mathbb{Q}_{>0}$ we let

$$z_X = \operatorname{span}(X) \in D$$

be the Heegner (or CM) point of discriminant m associated to X. We use the same symbol for the image of z_X in M. Note that the stabilizer Γ_X is finite.

A vector $X \in V$ of negative length $Q(X) = m \in \mathbb{Q}_{<0}$ defines a geodesic c_X in D via

$$c_X = \{z \in D \,:\, z \perp X\}.$$

We write $c(X) = \Gamma_X \backslash c_X$ for its image in M.

If $|m|/N$ is not a square in $\mathbb{Q}$, then $X^\perp$ is non-split over $\mathbb{Q}$ and the stabilizer $\overline{\Gamma}_X$ is infinite cyclic. On the other hand, if $|m|/N$ is a square, then $X^\perp$ is split and $\overline{\Gamma}_X$ is trivial. In the first case the geodesic $c(X)$ is closed, while in the second case $c(X)$ is an infinite geodesic (see also [Fun02, Lemma 3.6]).

In the case that $c(X)$ is an infinite geodesic, X is orthogonal to two isotropic lines $\ell_X = \operatorname{span}(Y)$ and $\widetilde{\ell}_X = \operatorname{span}(\widetilde{Y})$, where $Y, \widetilde{Y}$ are positively oriented such that $(X, Y, \widetilde{Y})$ is a positively oriented basis of V. Note that $\widetilde{\ell}_X = \ell_{-X}$.

For $h \in L'/L$ and $m \in \mathbb{Q}$ the group Γ acts on the set

$$L_{m,h} = \{X \in L + h \,:\, Q(X) = m\},$$

with finitely many orbits if $m \neq 0$. For $m > 0$ the set $L_{m,h}$ splits into a disjoint union of the subsets

$$L_{m,h}^+ = \left\{ X = \begin{pmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{pmatrix} \in L_{m,h} \,:\, x_3 > 0 \right\}$$

and

$$L_{m,h}^- = \left\{ X = \begin{pmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{pmatrix} \in L_{m,h} \,:\, x_3 < 0 \right\}.$$

Note that $x_3 = 0$ is not possible if $m > 0$.

2.2.4 Traces of CM values and cycle integrals

For $m \in \mathbb{Q}_{>0}$ and $h \in L'/L$ we define the modular trace function of a Γ-invariant function $F : \mathbb{H} \to \mathbb{C}$ by

$$\operatorname{tr}_F(m,h) = \sum_{X \in \Gamma \backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_X|} F(z_X),$$

where $\overline{\Gamma}_X$ denotes the stabilizer of X in $\overline{\Gamma} = \Gamma/\{\pm 1\}$. Similarly, we define the trace functions

$$\operatorname{tr}_F^+(m,h) = \sum_{X \in \Gamma \backslash L_{m,h}^+} \frac{1}{|\overline{\Gamma}_X|} F(z_X) \quad \text{and} \quad \operatorname{tr}_F^-(m,h) = \sum_{X \in \Gamma \backslash L_{m,h}^-} \frac{1}{|\overline{\Gamma}_X|} F(z_X),$$

such that $\operatorname{tr}_F(m,h) = \operatorname{tr}_F^+(m,h) + \operatorname{tr}_F^-(m,h)$.

For $m \in \mathbb{Q}_{<0}$ and $X \in L_{m,h}$ we define the cycle integral of a function $G : \mathbb{H} \to \mathbb{C}$, modular of weight $2k+2$ for Γ, along the geodesic $c(X)$ by

$$\mathcal{C}(G,X) = \int_{c(X)} G(z) Q_X^k(z) dz, \quad Q_X(z) = N(x_3 z^2 - 2x_2 z - x_1),$$

whenever the integral exists. Note that $Q_X(Mz) = j(M,z)^{-2} Q_{M^{-1}X}(z)$ where $j\,(M,z) = cz + d$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. The orientation of $c(X)$ is defined using an explicit parametrization as follows:

Since $Q(X) = m < 0$, there is some matrix $g \in \mathrm{SL}_2(\mathbb{R})$ such that

$$g^{-1}X = \sqrt{\frac{|m|}{N}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that the stabilizer $\overline{\Gamma}_X$ is either trivial or infinite cyclic. In the second case, the stabilizer of $g^{-1}X$ in $g^{-1}\overline{\Gamma}g$ is generated by some matrix $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$ with $\varepsilon > 1$. We can now parametrize $c(X)$ by $g.iy$ with $y \in (0,\infty)$ if $|m|/N$ is a square, and $y \in (1,\varepsilon^2)$ if $|m|/N$ is not a square. Note that

$$\frac{d}{dy} g.iy = i \cdot j(g,iy)^{-2},$$

and

$$Q_X(g.iy) = j(g,iy)^{-2} Q_{g^{-1}.X}(iy) = j(g,iy)^{-2} \left(-2\sqrt{|m|N} iy\right).$$

Writing $G_g = G|_{2k+2}g = j(g,z)^{-2k-2}G(gz)$ we find

$$\mathcal{C}(G,X) = \left(-2\sqrt{|m|N} i\right)^k i \int_0^\infty G_g(iy) y^k dy,$$

if $|m|/N$ is a square and similarly (i.e., with the integral from 1 to ε^2) if $|m|/N$ is not a square. Using the transformation behaviour of G it is easy to see that the right-hand side, and thus the implied orientation of $c(X)$, is independent of the choice of the matrix g. Finally, we define the trace of G for $m < 0$ by

$$\operatorname{tr}_G(m,h) = \sum_{X \in \Gamma \backslash L_{m,h}} \mathcal{C}(G,X).$$

2.2.5 A lattice corresponding to $\Gamma_0(N)$

A particularly interesting lattice is given by

$$L = \left\{ \begin{pmatrix} -b & -c/N \\ a & b \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}.$$

Its dual lattice is

$$L' = \left\{ \begin{pmatrix} -b/2N & -c/N \\ a & b/2N \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}.$$

We see that L'/L is isomorphic to $\mathbb{Z}/2N\mathbb{Z}$ with quadratic form $x \mapsto -x^2/4N$. Thus the level of L is $4N$. By a slight abuse of notation, we will view elements $h \in L'/L$ as elements of $\mathbb{Z}/2N\mathbb{Z}$ and vice versa. The group $\Gamma = \Gamma_0(N)$ acts on L and fixes the classes of L'/L, and the modular curve $\Gamma \backslash D$ equals $Y_0(N) = \Gamma_0(N) \backslash \mathbb{H}$ under the identification given in Section 2.2.1.

The significance of the lattice L lies in the fact that the elements

$$X = \begin{pmatrix} -b/2N & -c/N \\ a & b/2N \end{pmatrix} \in L'$$

correspond to integral binary quadratic forms

$$Q_X = \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix} X = \begin{pmatrix} aN & b/2 \\ b/2 & c \end{pmatrix}.$$

For $h \in \mathbb{Z}/2N\mathbb{Z}$ and $D \in \mathbb{Z}$ with $D \equiv h^2 \bmod 4N$ we let $\mathcal{Q}_{N,D,h}$ be the set of integral binary quadratic forms

$$Q(x,y) = aNx^2 + bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} aN & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

of discriminant $D = b^2 - 4Nac$ with $a,b,c \in \mathbb{Z}$ and $b \equiv h \bmod 2N$. We sometimes write $Q = [aN, b, c]$ for brevity. The group $\Gamma_0(N)$ acts on $\mathcal{Q}_{N,D,h}$ from the right by $QM = M^t QM$, with finitely many orbits if $D \neq 0$. The identification of X and Q_X is compatible with the corresponding actions of $\Gamma_0(N)$, in the sense that $Q_{gX} = Q_X g^{-1}$

for $g \in \Gamma_0(N)$. In particular, we have a bijection

$$\Gamma_0(N)\backslash L_{m,h} \cong \mathcal{Q}_{N,-4Nm,h}/\Gamma_0(N).$$

Let $Q = [aN, b, c] \in \mathcal{Q}_{N,D,h}$. If $D < 0$, then the order of the stabilizer of Q in $\Gamma_0(N)$ is finite, and there is an associated Heegner (or CM) point

$$z_Q = \frac{-b}{2Na} + i\frac{\sqrt{|D|}}{2N|a|} \in \mathbb{H},$$

which is characterized by $Q(z_Q, 1) = 0$. Further, $\mathcal{Q}_{N,D,h}$ splits into a disjoint union of the sets $\mathcal{Q}^+_{N,D,h}$ and $\mathcal{Q}^-_{N,D,h}$ of positive definite ($a > 0$) and negative definite ($a < 0$) quadratic forms, which correspond to the sets $L^+_{-D/4N,h}$ and $L^-_{-D/4N,h}$.

If $D > 0$, then Q is indefinite, and the stabilizer of Q in $\Gamma_0(N)/\{\pm 1\}$ is trivial if D is a square, and infinite cyclic otherwise. There is an associated geodesic in $\mathbb{H}$ given by

$$c_Q = \{z \in \mathbb{H} : aN|z|^2 + bx + c = 0\}.$$

These definitions of Heegner points and geodesics agree with the definitions made above, i.e., we have $c_X = c_{Q_X}$ and $z_X = z_{Q_X}$ for $X \in L_{m,h}$.

2.3 Harmonic Maass forms

In this section we introduce the notion of harmonic (weak) Maass forms, following Bruinier and Funke [BF04]. In contrast to the classical definition of Maass wave forms (see [Bum98]), which are required to be eigenforms of the invariant Laplace operator and to be square integrable with respect to the Petersson inner product, harmonic weak Maass forms should be harmonic with respect to the invariant Laplace operator but are allowed to grow linearly exponentially at the cusps. Since harmonic and square integrable automorphic forms are trivial (i.e., constant in weight 0 and vanishing identically in other weights), we will omit the word 'weak', and understand that a harmonic Maass form might grow linearly exponentially at the cusps.

We treat the theory of vector valued harmonic Maass forms of half-integral weight for the Weil representation associated to an even lattice L in some detail, and mention the necessary adjustments when working with scalar valued harmonic Maass forms of integral weight for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ at some places.

Throughout we let V be the rational quadratic space of signature $(1, 2)$ from Section 2.2. Further, we let $L \subset V$ be an arbitrary even lattice (unless otherwise specified) and Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which acts on L and fixes the classes of L'/L. We reserve the variable $\tau = u + iv \in \mathbb{H}$ for vector valued functions $f(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$, and $z = x + iy \in \mathbb{H}$ for scalar valued function $F(z) : \mathbb{H} \to \mathbb{C}$.

2.3.1 Harmonic Maass forms

Let $k \in \frac{1}{2} + \mathbb{Z}$ be half-integral. For $(M, \phi) \in \mathrm{Mp}_2(\mathbb{R})$ and a vector valued function

$$f = \sum_{h \in L'/L} f_h \mathfrak{e}_h : \mathbb{H} \to \mathbb{C}[L'/L]$$

we define the half-integral weight slash operator

$$(f|_{k,\rho_L}(M, \phi))(\tau) = \phi(\tau)^{-2k} \rho_L(M, \phi)^{-1} f(M\tau).$$

We let

$$\Delta_k = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$$

be the weight k hyperbolic Laplace operator. It acts component-wise on vector valued smooth functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and is invariant under the weight k slash action, that is,

$$\Delta_k (f|_{k,\rho_L}(M, \phi)) = (\Delta_k f)|_{k,\rho_L}(M, \phi)$$

for $(M, \phi) \in \mathrm{Mp}_2(\mathbb{R})$.

We recall the definition of a harmonic Maass form from [BF04].

Definition 2.3.1. A harmonic Maass form of weight $k \in \frac{1}{2} + \mathbb{Z}$ for ρ_L is a smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ with

1. $\Delta_k f = 0$,
2. $f|_{k,\rho_L}(M, \phi) = f$ for every $(M, \phi) \in \widetilde{\Gamma}$,
3. $f(\tau) = O(e^{Cv})$ as $v \to \infty$ for some constant $C > 0$.

We denote the space of such functions by H_{k,ρ_L}.

Scalar valued harmonic Maass forms of integral weight $k \in \mathbb{Z}$ for Γ are defined analogously, but the growth condition has to be checked at each cusp of Γ. The corresponding space is denoted by $H_k(\Gamma)$.

Remark 2.3.2. The action $\rho_L(Z)\mathfrak{e}_h = i\mathfrak{e}_{-h}$ of $Z = S^2$ in the Weil representation implies that $H_{k,\rho_L} = H_{k,\rho_L^*} = \{0\}$ unless $k \in \frac{1}{2} + \mathbb{Z}$, and that the components f_h of f satisfy the symmetry $f_{-h} = (-1)^{k+1/2} f_h$. Further, using the explicit formula for the action of the principal congruence subgroup $\Gamma(M)$ in the Weil representation, where M denotes the level of L for the moment, we see that the components f_h of a vector valued harmonic Maass form f are scalar valued harmonic Maass forms for $\Gamma(M)$.

Every harmonic Maass form $f \in H_{k,\rho_L}$ can be written as a sum $f = f^+ + f^-$ of a holomorphic and a non-holomorphic part having Fourier expansions of the form

$$\begin{aligned} f^+(\tau) &= \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_f^+(n,h) e(n\tau) \mathfrak{e}_h, \\ f^-(\tau) &= \sum_{h \in L'/L} \Big(c_f^-(0,h) v^{1-k} + \sum_{\substack{n \in \mathbb{Q} \setminus \{0\} \\ n \ll \infty}} c_f^-(n,h) H_k(4\pi n v) e(n\tau) \Big) \mathfrak{e}_h, \end{aligned} \tag{2.3.1}$$

with coefficients $c_f^\pm(n,h) \in \mathbb{C}$, where

$$H_k(w) = \int_{-w}^{\infty} e^{-t} t^{-k} dt.$$

For $w < 0$ we see that $H_k(w) = \Gamma(1-k, |w|)$ is an incomplete Gamma function, whereas for $w > 0$ the integral in $H_k(w)$ converges only for $k < 1$ and can be continued analytically in k to $\mathbb{C}$ in the same way as the Gamma function. Scalar valued harmonic Maass forms of integral weight $k \neq 1$ for Γ have a Fourier expansion of the above shape at each cusp of Γ, but without the sum over L'/L, of course. The finite sums

$$\begin{aligned} P_f^+(\tau) &= \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \leq 0}} c_f^+(n,h) e(n\tau) \mathfrak{e}_h, \\ P_f^-(\tau) &= \sum_{h \in L'/L} \Big(c_f^-(0,h) v^{1-k} + \sum_{\substack{n \in \mathbb{Q} \\ 0 < n \ll \infty}} c_f^-(n,h) H_k(4\pi n v) e(n\tau) \Big) \mathfrak{e}_h \end{aligned}$$

are called the holomorphic and the non-holomorphic principal part of f. Note that $f - P_f^+ - P_f^-$ is rapidly decreasing as $v \to \infty$.

Remark 2.3.3. Sometimes it is convenient to use a slightly different normalization of the Fourier expansion of the non-holomorphic part of a harmonic Maass form, given by

$$\begin{aligned} f^-(\tau) = \sum_{h \in L'/L} \Big(c_f^-(0,h) v^{1-k} &+ \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c_f^-(n,h) v^{1-k} \beta_k(-4\pi n v) e(n\tau) \\ &+ \sum_{\substack{n \in \mathbb{Q} \\ 0 < n \ll \infty}} c_f^-(n,h) v^{1-k} \beta_k^c(-4\pi n v) e(n\tau) \Big) \mathfrak{e}_h, \end{aligned} \tag{2.3.2}$$

with coefficients $c_f^\pm(n,h) \in \mathbb{C}$, where

$$\beta_k(w) = \int_1^\infty e^{-wt} t^{-k} dt = w^{k-1} \Gamma(1-k, w), \qquad \beta_k^c(w) = \int_0^1 e^{-wt} t^{-k} dt.$$

In fact, all the special functions appearing in the two normalizations of the Fourier expansions are special cases of Whittaker functions, compare the expansion given in (2.3.12).

We let H^+_{k,ρ_L} be the subspace of harmonic Maass forms f whose non-holomorphic principal part $P_f^-(\tau)$ vanishes, which means that the non-holomorphic part f^- of f is rapidly decreasing as $v \to \infty$. Further, we let $M^!_{k,\rho_L}$ be the space of weakly holomorphic modular forms, consisting of the forms in H_{k,ρ_L} which are holomorphic on $\mathbb{H}$, i.e., $f^- = 0$. We let M_{k,ρ_L} be the space of holomorphic modular forms ($c_f^+(n,h) = 0$ for $n < 0$) and S_{k,ρ_L} the space of cusp forms ($c_f^+(n,h) = 0$ for $n \leq 0$). We have the inclusions

$$S_{k,\rho_L} \subseteq M_{k,\rho_L} \subseteq M^!_{k,\rho_L} \subseteq H^+_{k,\rho_L} \subseteq H_{k,\rho_L}.$$

Example 2.3.4. 1. The non-holomorphic Eisenstein series

$$E_2^*(z) = -\frac{3}{\pi y} + 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) e(nz)$$

is a scalar valued harmonic Maass form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$.

2. Zagier's non-holomorphic Eisenstein series

$$E^*_{3/2}(\tau) = \sum_{d=0}^{\infty} H(d)e(d\tau) + \frac{1}{16\pi\sqrt{v}} \sum_{n\in\mathbb{Z}} \beta_{3/2}(4\pi n^2 v)e(-n^2\tau),$$

with $H(0) = -\frac{1}{12}$ and the Hurwitz class numbers

$$H(d) = \sum_{Q \in \mathcal{Q}_{-d}/\,\mathrm{SL}_2(\mathbb{Z})} \frac{1}{|\,\mathrm{PSL}_2(\mathbb{Z})_Q|},$$

is a scalar valued harmonic Maass form of weight $3/2$ for $\Gamma_0(4)$ satisfying the Kohnen plus space condition, meaning that its Fourier expansion is supported on indices with $-d \equiv 0, 1 \bmod 4$ (see [Zag75]). It can be viewed as a vector valued harmonic Maass form for the Weil representation ρ_L of the lattice L defined in Section 2.2.5 (see Theorem 2.3.15 below) or as a harmonic Maass Jacobi form of weight 2 and index 1 (see Theorem 2.3.11 below).

2.3.2 Differential operators

For $\tau = u + iv \in \mathbb{H}$ we let

$$\frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial\bar{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right),$$

be the usual Wirtinger derivatives. Recall that a function f is holomorphic if and only if $\frac{\partial}{\partial\bar{\tau}}f = 0$. For $k \in \frac{1}{2}\mathbb{Z}$ the Maass lowering and raising operators are defined by

$$L_k = -2iv^2\frac{\partial}{\partial\bar{\tau}}, \qquad R_k = 2i\frac{\partial}{\partial\tau} + kv^{-1},$$

and they act component-wise on functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$. These operators commute with the slash operator, in the sense that

$$(L_kf)|_{k-2,\rho_L}(M,\phi) = L_k(f|_{k,\rho_L}(M,\phi)), \qquad (R_kf)|_{k+2,\rho_L}(M,\phi) = R_k(f|_{k,\rho_L}(M,\phi))$$

for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{R})$. In particular, if f is modular of weight k for ρ_L then L_kf and R_kf are modular of weight $k-2$ and $k+2$ for ρ_L, respectively.

The lowering and raising operators are related to the weighted Laplace operator by

$$-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k. \tag{2.3.3}$$

This implies the commutation relations

$$R_k\Delta_k = (\Delta_{k+2} - k)R_k, \tag{2.3.4}$$
$$\Delta_{k-2}L_k = L_k(\Delta_k + 2 - k). \tag{2.3.5}$$

We also define iterated versions of the lowering and raising operators by

$$L_k^n = L_{k-2(n-1)} \circ \cdots \circ L_{k-2} \circ L_k, \qquad R_k^n = R_{k+2(n-1)} \circ \cdots \circ R_{k+2} \circ R_k.$$

For $n = 0$ we set $L_k^0 = R_k^0 = \mathrm{id}$. Using (2.3.4) and (2.3.5) inductively one can find many interesting commutation relations between the iterated lowering and raising operators and the weighted Laplacian. We collect some identities for later use.

Lemma 2.3.5. *We have the following relations.*

1. *For $k \in \mathbb{Z}_{\geq 0}$ and $\ell = 0, \ldots, k$ we have*

$$\Delta_{-2\ell}R_{-2k}^{k-\ell} = R_{-2k}^{k-\ell}\left(\Delta_{-2\ell} - (k-\ell)(k+\ell+1)\right).$$

2. *If k is even then*

$$\Delta_{1/2-k}L_{1/2}^{k/2} = L_{1/2}^{k/2}\left(\Delta_{1/2} + \frac{k}{4}(k+1)\right),$$
$$\Delta_{3/2+k}R_{3/2}^{k/2} = R_{3/2}^{k/2}\left(\Delta_{3/2} + \frac{k}{4}(k+1)\right),$$

3. If k is odd we have

$$\Delta_{1/2-k} L_{3/2}^{(k+1)/2} = L_{3/2}^{(k+1)/2}\left(\Delta_{3/2} + \frac{k}{4}(k+1)\right),$$
$$\Delta_{3/2+k} R_{1/2}^{(k+1)/2} = R_{1/2}^{(k+1)/2}\left(\Delta_{1/2} + \frac{k}{4}(k+1)\right).$$

An important tool in the theory of harmonic Maass forms is the antilinear differential operator

$$\xi_k f = v^{k-2}\overline{L_k f(\tau)} = R_{-k} v^k \overline{f(\tau)} = 2iv^k \overline{\frac{\partial}{\partial \bar{\tau}} f(\tau)}.$$

It defines a surjective map

$$\xi_k : H_{k,\rho_L} \to M^!_{2-k,\rho_L^*}.$$

Note that a harmonic Maass form f lies in H^+_{k,ρ_L} if and only if $\xi_k f$ is a cusp form.

There is another important differential operator which acts on harmonic Maass forms of integral weight $k \in \mathbb{Z}$ with $k \leq 1$. In [BOR08] the authors introduced the linear differential operator

$$D^{1-k} = \left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{1-k}$$

which defines a map

$$D^{1-k} : H_k(\Gamma) \to M^!_{2-k}(\Gamma).$$

It acts on the Fourier expansion of F by

$$D^{1-k} F = D^{1-k} F^+ = \sum_{n \gg -\infty} c_F^+(n) n^{1-k} e(nz).$$

Bol's identity states that D^{1-k} is related to the iterated raising operator by

$$D^{1-k} = \frac{1}{(-4\pi)^{1-k}} R_k^{1-k}.$$

By applying the Hecke bound to the cusp form $D^{1-k}F$ we immediately obtain that the coefficients $c_F^+(n)$ of a harmonic Maass form F grow polynomially in n if the holomorphic principal parts of F vanish at all cusps. The analogous result in the half-integral weight setting is quite difficult to prove, and involves the explicit construction of a basis of the space of harmonic weak Maass forms and a detailed study of their Fourier coefficients, see Section 2.3.8 below. Unfortunately, this differential operator does not have a half-integral weight counterpart.

2.3.3 The regularized inner product and integral formulas

Let $k \in \frac{1}{2} + \mathbb{Z}$. The regularized inner product of $f \in M^!_{k,\rho_L}$ and $g \in S_{k,\rho_L}$ is defined by

$$(f,g)^{\mathrm{reg}} = \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f(\tau), g(\tau)\rangle v^k \frac{du\,dv}{v^2}, \tag{2.3.6}$$

where

$$\mathcal{F}_T = \{\tau = u + iv \in \mathbb{H} : |u| \leq \tfrac{1}{2}, |\tau| \geq 1, v \leq T\}$$

is a truncated fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. For $k = 1/2$ the regularized inner product also converges for $g \in M_{1/2,\rho_L}$. If the integral exists without regularization, e.g., if both f and g are cusp forms, then $(f,g)^{\mathrm{reg}}$ agrees with the usual Petersson inner product (f,g).

An application of Stokes' theorem (compare Proposition 3.5 in [BF06]) shows that for $f \in M^!_{k,\rho_L}$ with coefficients $c_f(n,h)$, and $\widetilde{g} \in H_{2-k,\rho_L^*}$ with holomorphic coefficients $c^+_{\widetilde{g}}(n,h)$ and $\xi_{2-k}\widetilde{g} = g \in S_{k,\rho_L}$ (or $g \in M_{1/2,\rho_L}$ if $k = 1/2$), the regularized inner product can be evaluated as

$$(f,g)^{\mathrm{reg}} = (f, \xi_{2-k}\widetilde{g})^{\mathrm{reg}} = \sum_{h\in L'/L} \sum_{n\in\mathbb{Q}} c_f(n,h) c^+_{\widetilde{g}}(-n,h). \tag{2.3.7}$$

The regularized inner product of scalar valued forms of integral weight for Γ is defined analogously, and the above evaluation works in the same way, but now the Fourier coefficients at all cusps of Γ contribute on the right-hand side.

The above formula can be employed to show the following useful lemma.

Lemma 2.3.6. *Let f be a harmonic Maass form of weight $2-k$ for ρ_L or ρ_L^* whose principal part vanishes and which maps to a cusp form under ξ_{2-k} (or a holomorphic modular form if $k = 1/2$). Then f is a cusp form.*

Proof. Suppose that $g = \xi_{2-k}f$ is a cusp form. By (2.3.7) we see that $(g,g) = (g,g)^{\mathrm{reg}} = (g, \xi_{2-k}f)^{\mathrm{reg}} = 0$, so $g = 0$. This means that f is holomorphic, hence a cusp form. For $k = 1/2$ the inner product (g,g) also converges if $g \in M_{1/2,\rho_L}$, so the same argument works in this case. $\square$

Now let $k \in \mathbb{Z}$. For the computation of the Fourier expansion of the theta lifts studied in this work we will need integral formulas which allow us to move differential operators from one function in the integral to another. The formulas are simple consequences of Stokes' theorem, which states that for a smooth $(n-1)$-form ω on some n-dimensional compact oriented smooth manifold X with boundary ∂X we have

$$\int_X d\omega = \int_{\partial X} \omega,$$

where d denotes the usual exterior derivative. Let

$$d\mu(z) = \frac{dx \wedge dy}{y^2} = \frac{i}{2} \cdot \frac{dz \wedge d\bar{z}}{y^2}$$

be the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}$.

Lemma 2.3.7. *Let $k \in \mathbb{Z}$, and let $F, G : \mathbb{H} \to \mathbb{C}$ be smooth functions which transform like $F|_{2-k}M = F$ and $G|_kM = G$ for all $M \in \Gamma$. Then for $T > 1$ we have*

$$\int_{M_T} \overline{\xi_{2-k}F(z)}G(z)y^k d\mu(z) + \int_{M_T} F(z)\overline{\xi_k G(z)}y^{2-k} d\mu(z) = -\int_{\partial M_T} F(z)G(z)dz.$$

Proof. Writing $d = \partial + \bar{\partial}$ with $\partial F = \frac{\partial}{\partial z}F(z)dz$ and $\bar{\partial}F = \frac{\partial}{\partial \bar{z}}F(z)d\bar{z}$, and

$$\begin{aligned} d\big(F(z)G(z)dz\big) = \bar{\partial}\big(F(z)G(z)dz\big) &= \left(\frac{\partial}{\partial \bar{z}}\big(F(z)G(z)\big)\right) d\bar{z} \wedge dz \\ &= -\left(\overline{\xi_{2-k}F(z)}G(z)y^k + F(z)\overline{\xi_k G(z)}y^{2-k}\right) d\mu(z), \end{aligned}$$

the formula follows from Stokes' theorem applied to the smooth 1-form $F(z)G(z)dz$ on M_T. ☐

Remark 2.3.8. Let F and G be as in the lemma above. For each cusp $\ell \in \Gamma\backslash \operatorname{Iso}(V)$ of Γ, choose a matrix $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ with $\sigma_\ell \infty = \ell$ and let α_ℓ denote the width of ℓ. Write $F_\ell = F|_{2-k}\sigma_\ell$ and $G_\ell = G|_k\sigma_\ell$ for brevity. The integral over the boundary can then be written as

$$\begin{aligned} \int_{\partial M_T} F(z)G(z)dz &= \sum_{\ell \in \Gamma\backslash \operatorname{Iso}(V)} \int_{\partial \mathcal{F}_T^{\alpha_\ell}} F_\ell(z)G_\ell(z)dz \\ &= -\sum_{\ell \in \Gamma\backslash \operatorname{Iso}(V)} \int_{iT}^{\alpha_\ell + iT} F_\ell(z)G_\ell(z)dz, \end{aligned}$$

since in the integral over $\partial\mathcal{F}_T^{\alpha_\ell}$ all terms but the horizontal boundary piece cancel out due to the modularity of the integrand. The minus sign comes from the fact that we integrate over $\partial\mathcal{F}_T^{\alpha_\ell}$ in counter-clockwise direction.

Using $\Delta_k = -\xi_{2-k}\xi_k$ we obtain the following result.

Lemma 2.3.9. *Let $F, G : \mathbb{H} \to \mathbb{C}$ be smooth functions which transform like $F|_kM = F$*

and $G|_k M = G$ for all $M \in \Gamma$. Then for $T > 1$ we have

$$\begin{aligned}
&\int_{M_T} \overline{\Delta_k F(z)} G(z) y^k d\mu(z) - \int_{M_T} \overline{F(z)} \Delta_k G(z) y^k d\mu(z) \\
&= \int_{\partial M_T} \big(\xi_k F(z)\big) G(z) dz - \overline{\int_{\partial M_T} F(z) \big(\xi_k G(z)\big) dz}.
\end{aligned}$$

2.3.4 Jacobi forms

We now define holomorphic and skew-holomorphic Jacobi forms of integral weight and explain their connection to vector valued modular forms of half-integral weight. The standard reference on Jacobi forms is the book [EZ85] by Eichler and Zagier.

Throughout this section we let L be the lattice related to $\Gamma_0(N)$ which was defined in Section 2.2.5. Equivalently, we could take the one dimensional negative definite lattice $\mathbb{Z}$ with the quadratic form $n \mapsto -Nn^2$ instead of L, since its discriminant group is also isomorphic to $\mathbb{Z}/2N\mathbb{Z}$ with the finite quadratic form $x \mapsto -x^2/4N$.

The group $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{Z}^2$ (with elements of $\mathbb{Z}^2$ viewed as row vectors) with group law

$$[M, X] \cdot [M', X'] = [MM', XM' + X']$$

acts on holomorphic functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned}
&\phi\Big|_{k,N} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [\lambda, \mu]\right](\tau, z) \qquad (2.3.8) \\
&= (c\tau + d)^{-k} e^N\left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right),
\end{aligned}$$

where $e(x) = e^{2\pi i x}$ and $e^N(x) = e^{2\pi i N x}$ for $x \in \mathbb{C}, N \in \mathbb{Z}$.

Definition 2.3.10. A Jacobi form of weight $k \in \mathbb{Z}$ and index N is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ with

1. $\phi|_{k,N}[M, X] = \phi$ for every $[M, X] \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.
2. The function $\phi(\tau, z)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{D, r \in \mathbb{Z}, D \leq 0 \\ r^2 \equiv D(4N)}} c_\phi(D, r) q^{\frac{r^2 - D}{4N}} \zeta^r, \qquad (q = e(\tau),\ \zeta = e(z)),$$

   with coefficients $c_\phi(D, r) \in \mathbb{C}$.

If $c_\phi(D, r) = 0$ whenever $D = 0$, we call ϕ a Jacobi cusp form. The corresponding spaces of Jacobi forms are denoted by $J_{k,N}$ and $J^{\mathrm{cusp}}_{k,N}$.

The invariance property

$$\phi(\tau, z + \lambda\tau + \mu) = \phi(\tau, z), \qquad [\lambda, \mu] \in \mathbb{Z}^2,$$

of a Jacobi form implies that for fixed D the coefficient $c_\phi(D, r)$ only depends on the residue class of $r \bmod 2N$. Thus we can write

$$\phi(\tau, z) = \sum_{r(2N)} f_r(\tau)\vartheta_r(\tau, z),$$

where

$$f_r(\tau) = \sum_{\substack{D\in\mathbb{Z}, D\leq 0\\ D\equiv r^2(4N)}} c_\phi(D, r)q^{-\frac{D}{4N}}, \qquad \vartheta_r(\tau, z) = \sum_{\substack{m\in\mathbb{Z}\\ m\equiv r(2N)}} q^{\frac{m}{4N}}\zeta^m.$$

This is the so-called theta decomposition of the Jacobi form ϕ. The transformation laws of ϕ and of the theta functions $\vartheta_r(\tau, z)$, which can be derived using Poisson summation, together imply a certain transformation behaviour of the functions $f_r(\tau)$. More precisely, we obtain:

Theorem 2.3.11 ([EZ85], Theorem 5.1)**.** *The map*

$$\phi \mapsto \sum_{r(2N)} f_r(\tau)\mathfrak{e}_r$$

yields isomorphisms

$$J_{k,N} \cong M_{k-1/2,\rho_L} \quad \text{and} \quad J_{k,N}^{\text{cusp}} \cong S_{k-1/2,\rho_L}.$$

Next, we will see that the space $M_{k+1/2,\rho_L^*}$ of holomorphic modular forms for the dual Weil representation is isomorphic to the space $J_{k,N}^*$ of skew-holomorphic Jacobi forms, whose definition we now recall from [Sko90]. We define a modified slash operation $|_{k,N}^*$ of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ by replacing the factor $(c\tau + d)^{-k}$ in (2.3.8) by

$$|c\tau + d|^{-1}(c\bar{\tau} + d)^{-k+1}.$$

Definition 2.3.12. A skew-holomorphic Jacobi form of weight $k \in \mathbb{Z}$ and index N is a smooth function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ with

1. $\phi|_{k,N}^*[M, X] = \phi$ for every $[M, X] \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.
2. The function $\phi(\tau, z)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{D,r\in\mathbb{Z}, D\geq 0\\ D\equiv r^2(4N)}} c_\phi(D, r)e\left(\frac{D}{2N}iv\right)q^{\frac{r^2-D}{4N}}\zeta^r, \qquad (q = e(\tau),\ \zeta = e(z)),$$

with coefficients $c_\phi(D,r) \in \mathbb{C}$.

If $c_\phi(D,r) = 0$ whenever $D = 0$, we call ϕ a skew-holomorphic Jacobi cusp form. The corresponding spaces of skew-holomorphic Jacobi forms are denoted by $J^*_{k,N}$ and $J^{*,\mathrm{cusp}}_{k,N}$.

Similarly as above, each $\phi \in J^*_{k,N}$ has a theta decomposition

$$\phi(\tau,z) = \sum_{r(2N)} f_r(\tau)\vartheta_r(\tau,z)$$

where $\vartheta_r(\tau,z)$ is the same theta function as above, and

$$f_r(\tau) = \sum_{\substack{D\in\mathbb{Z}, D\geq 0\\ D\equiv r^2(4N)}} c_\phi(D,r)e\left(-D\bar{\tau}\right).$$

We obtain

Theorem 2.3.13 ([EZ85], Theorem 5.1)**.** *The map*

$$\phi \mapsto \sum_{r(2N)} \overline{f_r(\tau)}\mathfrak{e}_r$$

yields isomorphisms

$$J^*_{k,N} \cong M_{k-1/2,\rho^*_L} \quad \text{and} \quad J^{*,\mathrm{cusp}}_{k,N} \cong S_{k-1/2,\rho^*_L}.$$

This connection enables us to carry over results from the theory of Jacobi forms to vector valued modular forms for the Weil representation.

Theorem 2.3.14 ([EZ85], Theorem 5.7; [SZ88], p. 130)**.** *We have*

$$M_{1/2,\rho_L} \cong J_{1,N} = \{0\}$$

and

$$\dim(M_{1/2,\rho^*_L}) = \dim(J^*_{1,N}) = \frac{1}{2}(\sigma_0(N) + \delta(N = \square)),$$

for all N, where $\sigma_0(N) = \sum_{d|N} 1$ is the number of positive divisors of N, and $\delta(N = \square)$ equals 1 if N is a square, and 0 otherwise.

We remark that in [SZ88], p. 130, the authors also construct a basis of $J^*_{1,N}$, which consists of certain theta series. Analogously, we will construct a basis of $M_{1/2,\rho^*_L}$ consisting of unary theta series in Lemma 2.3.17.

Theorem 2.3.15 ([EZ85], Theorem 5.6)**.** *Let $k \in \frac{1}{2} + \mathbb{Z}$. Then the map*

$$\sum_{h\in L'/L} f_h(\tau)\mathfrak{e}_h \mapsto \sum_{h\in L'/L} f_h(4N\tau),$$

sends a holomorphic vector valued modular form of weight k for ρ_L (resp. ρ_L^*) to a scalar valued holomorphic modular form of weight k for $\Gamma_0(4N)$ whose n-th Fourier coefficient vanishes unless $-n$ (resp. n) is a square mod $4N$. If $N = 1$, or if $N = p$ is a prime and $k + \frac{1}{2}$ is even (resp. odd), this map is an isomorphism between the two spaces of holomorphic modular forms.

As a special case, let $N = 1$ and let $M_k^+(\Gamma_0(4))$ be the space of scalar valued holomorphic modular forms of weight k for $\Gamma_0(4)$ satisfying the Kohnen plus space condition $c_f(n) = 0$ unless $(-1)^{k-\frac{1}{2}}n \equiv 0, 1 \bmod 4$. Then the map

$$f_0(\tau)\mathfrak{e}_0 + f_1(\tau)\mathfrak{e}_1 \mapsto f_0(4\tau) + f_1(4\tau)$$

defines isomorphisms $M_{k,\rho_L} \cong M_k^+(\Gamma_0(4))$ if $k + \frac{1}{2}$ is even, and $M_{k,\rho_L^*} \cong M_k^+(\Gamma_0(4))$ if $k + \frac{1}{2}$ is odd.

2.3.5 Atkin-Lehner and level raising operators

Let $k \in \frac{1}{2} + \mathbb{Z}$. The orthogonal group $O(L'/L)$ of the finite quadratic module L'/L acts on vector valued modular forms

$$f(\tau) = \sum_{h \in L'/L} f_h(\tau)\mathfrak{e}_h$$

for ρ_L or ρ_L^* by

$$f^\sigma(\tau) = \sum_{h \in L'/L} f_h(\tau)\mathfrak{e}_{\sigma(h)},$$

where $\sigma \in O(L'/L)$. Now let L be the lattice related to $\Gamma_0(N)$ from Section 2.2.5, such that $L'/L \cong \mathbb{Z}/2N\mathbb{Z}$, and vector valued modular forms can be identified with Jacobi forms. Then the elements of $O(L'/L)$ correspond to the Atkin-Lehner involutions from the theory of Jacobi forms, see [EZ85], Theorem 5.2. These operators in turn correspond to the exact divisors $c \mid\mid N$ (i.e., $c \mid N$ and $(c, N/c) = 1$). The automorphism σ_c corresponding to c is defined by the equations

$$\sigma_c(h) \equiv -h \ (2c) \quad \text{and} \quad \sigma_c(h) \equiv h \ (2N/c) \tag{2.3.9}$$

for $h \in \mathbb{Z}/2N\mathbb{Z}$. Note that the Atkin-Lehner involutions only permute the components of a vector valued modular form.

For each positive integer d there is an operator U_d which maps Jacobi forms of index N to forms of index Nd^2, see [EZ85], Section 4. Using the isomorphism $J_{k,N} \cong M_{k-1/2,\rho_L}$ it yields an operator which maps modular forms for the Weil representation of the lattice L of level $4N$ to forms for the lattice L of level $4Nd^2$. Its action on the Fourier expansion of a holomorphic modular form

$$f(\tau) = \sum_{h(2N)} \sum_{n \in \mathbb{Q}} c_f(n,h) e(n\tau)\, \mathfrak{e}_h$$

for ρ_L is given by

$$(f|U_d)(\tau) = \sum_{\substack{h(2Nd^2) \\ h \equiv 0(d)}} \sum_{n \in \mathbb{Q}} c_f(n/d^2, h/d) e(n\tau)\, \mathfrak{e}_h. \tag{2.3.10}$$

In particular, U_d only distributes the components of f in a certain way, but does not change the set of Fourier coefficients of f.

The Atkin-Lehner and U_d operators also act on harmonic Maass forms by the same formulas as above, and they commute with the ξ-operator.

We now introduce Atkin-Lehner operators on scalar valued modular forms. Let $k \in \mathbb{Z}$, let N be a positive integer, and let $\Gamma = \Gamma_0(N)$. For each exact divisor $d \mid\mid N$ (meaning $d \mid N$ and $(d, N/d) = 1$) choose a matrix

$$W_d^N = \begin{pmatrix} d\alpha & \beta \\ N\gamma & d\delta \end{pmatrix},$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that W_d^N has determinant d. The operator

$$f \mapsto f|_k W_d^N$$

is independent of the choice of the parameters and defines an involution on $H_k(\Gamma_0(N))$, which is called the Atkin-Lehner involution corresponding to d. For two exact divisors d, d' of N we have

$$F|_k W_d^N | W_{d'}^N = F|_k W_{d*d'}^N,$$

where $d * d' = \frac{dd'}{(d,d')^2}$ is again an exact divisor of N. In particular, the Aktin-Lehner involutions form a finite abelian group which is isomorphic to the set of all exact divisors with the $*$ multiplication.

2.3.6 Unary theta functions

Let L be the lattice related to $\Gamma_0(N)$ from Section 2.2.5. We define the unary theta functions

$$\theta_{1/2,N}(\tau) = \sum_{h(2N)} \sum_{\substack{b \in \mathbb{Z} \\ b \equiv h(2N)}} e(b^2\tau/4N)\mathfrak{e}_h \qquad \text{and} \qquad \theta_{3/2,N}(\tau) = \sum_{h(2N)} \sum_{\substack{b \in \mathbb{Z} \\ b \equiv h(2N)}} be(b^2\tau/4N)\mathfrak{e}_h. \tag{2.3.11}$$

They are holomorphic vector valued modular forms of weight 1/2 and 3/2 for ρ_L^*, which follows from Theorem 2.1.1 by noting that they are the theta functions associated to the 1-dimensional positive definite lattice $\mathbb{Z}$ with the quadratic form $x \mapsto Nx^2$ and the polynomials 1 and x. By Theorem 2.3.13 one can view $\theta_{1/2}$ and $\theta_{3/2}$ as skew-holomorphic Jacobi forms of weight 1 and 2.

Definition 2.3.16. The space of unary theta functions of weight 1/2 for ρ_L^* is defined as

$$\sum_{d^2|N} \sum_{c||\frac{N}{d^2}} \mathbb{C}\; \theta^{\sigma_c}_{1/2,N/d^2}|U_d\,.$$

The space of unary theta functions of weight 3/2 for ρ_L^* is defined analogously.

We defined the spaces of unary theta functions only for the dual Weil representation ρ_L^* since there are no similar theta functions for ρ_L.

We now show that the space of unary theta functions of weight 1/2 agrees with the whole space $M_{1/2,\rho_L^*}$.

Lemma 2.3.17. *Let $\mathcal{D}(N)$ be the set of all positive divisors of N modulo the equivalence relation $c \sim N/c$. Then the theta functions*

$$\theta^{\sigma_{c/(c,N/c)}}_{1/2,N/(c,N/c)^2}|U_{(c,N/c)}, \quad c \in \mathcal{D}(N),$$

form a basis of $M_{1/2,\rho_L^*}$.

Proof. Using the dimension formula for $J^*_{1,N} \cong M_{1/2,\rho_L^*}$ from Theorem 2.3.14, we obtain that the number of given theta functions agrees with the dimension of $M_{1/2,\rho_L^*}$. On the other hand, by looking at the constant terms and the coefficients at $q^{d^2/4N}\mathfrak{e}_d$ for $d \in \mathcal{D}(N)$, we see that the given functions are linearly independent. $\square$

2.3.7 Maass Poincaré series

We construct some explicit examples of vector valued harmonic Maass forms using Maass Poincaré series, with special focus on the case $k = 1/2$. In the next section, we will estimate the growth of the coefficients of such series, yielding an estimate for the growth of the coefficients of arbitrary harmonic Maass forms.

We let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the usual Whittaker functions as defined in [AS64], Chapter 13. For $k \in \frac{1}{2}\mathbb{Z}, n \in \mathbb{Q}, v > 0$ and $s \in \mathbb{C}$ we set

$$\mathcal{M}_{n,k}(v,s) = \begin{cases} \Gamma(2s)^{-1}(4\pi|n|v)^{-k/2} M_{\operatorname{sgn}(n)k/2,s-1/2}(4\pi|n|v), & n \neq 0, \\ v^{s-k/2}, & n = 0, \end{cases}$$

and

$$\mathcal{W}_{n,k}(v,s) = \begin{cases} \Gamma(s+\mathrm{sgn}(n)k/2)^{-1}|n|^{k/2-1}(4\pi v)^{-k/2}W_{\mathrm{sgn}(n)k/2,s-1/2}(4\pi|n|v), & n \neq 0, \\ \frac{(4\pi)^{1-k}}{(2s-1)\Gamma(s-k/2)\Gamma(s+k/2)}v^{1-s-k/2}, & n = 0. \end{cases}$$

The normalization is taken from [JKK13] and is chosen such that the Fourier expansion of the Maass Poincaré series studied below takes a simpler form. We will be particularly interested in the point $s = 1 - k/2$ for $k \leq 1/2$. We abbreviate

$$\mathcal{M}_{n,k}(v) = \mathcal{M}_{n,k}(v, 1-k/2), \qquad \mathcal{W}_{n,k}(v) = \mathcal{W}_{n,k}(v, 1-k/2).$$

Then we have

$$\mathcal{M}_{n,k}(v) = e^{-2\pi ny} \begin{cases} (-1)^k[\Gamma(1-k)^{-1}\Gamma(1-k,-4\pi nv) - 1], & n > 0, \\ 1 - \Gamma(1-k)^{-1}\Gamma(1-k,-4\pi nv), & n < 0, \\ v^{1-k}, & n = 0, \end{cases}$$

and

$$\mathcal{W}_{n,k}(v) = e^{-2\pi ny} \begin{cases} n^{k-1}, & n > 0, \\ |n|^{k-1}\Gamma(1-k)^{-1}\Gamma(1-k,-4\pi nv), & n < 0, \\ \Gamma(2-k)^{-1}(4\pi)^{1-k}, & n = 0. \end{cases}$$

The Fourier expansion of a harmonic Maass form $f \in H_{k,\rho_L}$ can also be written in the form

$$f(\tau) = \sum_{h \in L'/L} \sum_{n \in \mathbb{Q}} a_f(n,h)\mathcal{M}_{n,k}(v)e(nu)\mathfrak{e}_h + \sum_{h \in L'/L} \sum_{n \in \mathbb{Q}} b_f(n,h)\mathcal{W}_{n,k}(v)e(nu)\mathfrak{e}_h \quad (2.3.12)$$

with coefficients $a(n,h), b(n,h) \in \mathbb{C}$. The translation between the normalizations in (2.3.12) and (2.3.1) is simple. Note that the condition $f(\tau) = O(e^{Cv})$ as $v \to \infty$ and the asymptotic behaviour of the Whittaker functions imply that the first sum over n is finite. This normalization of the Fourier expansion emphasizes the splitting of f into an increasing and a decreasing part (as $v \to \infty$), and is more convenient for the construction of Maass Poincaré series.

Let $k \in 1/2 + \mathbb{Z}$ with $k \leq 1/2$. For $h \in L'/L$ and $m \in \mathbb{Q}$ we consider the non-holomorphic Maass Poincaré series

$$P_{k,m,h}(\tau,s) = \frac{1}{2} \sum_{(M,\phi) \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} [\mathcal{M}_{m,k}(v,s)e(mu)\mathfrak{e}_h] \,|_{k,\rho_L}(M,\phi).$$

It converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1$, and it has weight k with respect to ρ_L. Maass-Poincaré series for the dual Weil representation ρ_L^* are defined analogously, and everything we say in this section remains true for these series as well. The Poincaré series satisfies the Laplace equation

$$\Delta_k P_{k,m,h}(\tau,s) = (s-k/2)(1-k/2-s)P_{k,m,h}(\tau,s), \tag{2.3.13}$$

which follows by a direct calculation from the defining differential equation of the M-Whittaker function. This implies that $P_{k,m,h}(\tau,s)$ is real analytic in $\tau\in\mathbb{H}$. Note that for $k\leq -1/2$ the Poincaré series converges at $s=1-k/2$, giving a harmonic function.

The Fourier expansion of $P_{k,m,h}(\tau,s)$ has been computed in several recent works, for example in [Bru02, Theorem 1.9], [DIT11, Proposition 2] and [JKK13, Theorem 3.2]. The result is as follows.

Proposition 2.3.18. *For* $\mathrm{Re}(s)>1$ *the Poincaré series* $P_{k,m,h}(\tau,s)$ *has the Fourier expansion*

$$P_{k,m,h}(\tau,s) = \mathcal{M}_{m,k}(v,s)e(mu)(\mathfrak{e}_h+\mathfrak{e}_{-h}) + \sum_{h'\in L'/L}\sum_{n\in\mathbb{Q}} b_{k,m,h}(n,h',s)\,\mathcal{W}_{n,k}(v,s)e(nu)\mathfrak{e}_{h'}$$

where the Fourier coefficients $b_{k,m,h}(n,h',s)$ *are given by*

$$2\pi\sum_{c\neq 0}H_c(h,m,h',n)\times\begin{cases}|mn|^{(1-k)/2}J_{2s-1}\big(4\pi\sqrt{|mn|}/|c|\big), & mn>0,\\ |mn|^{(1-k)/2}I_{2s-1}\big(4\pi\sqrt{|mn|}/|c|\big), & mn<0,\\ 2^{k-1}\pi^{s+k/2-1}|m+n|^{s-k/2}|c|^{1-2s}, & mn=0, m+n\neq 0,\\ 2^{2k-2}\pi^{k-1}\Gamma(2s)|2c|^{1-2s}, & m=n=0.\end{cases}$$

Here

$$H_c(h,m,h',n) = \frac{e(-\operatorname{sgn}(c)k/4)}{|c|}\sum_{\substack{d(c)^*\\ \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})}}\langle\rho_L^{-1}(((\begin{smallmatrix}a&b\\c&d\end{smallmatrix}),\sqrt{c\tau+d}))\mathfrak{e}_h,\mathfrak{e}_{h'}\rangle e\left(\frac{ma+nd}{c}\right)$$

is a Kloosterman sum, and I_{2s-1} *and* J_{2s-1} *are the usual Bessel functions. The Fourier expansion converges for* $\mathrm{Re}(s)>1/2$.

In the following, we restrict to the case $k=1/2$ for simplicity, but the results remain true for $k\leq -1/2$, with much simpler proofs since $P_{k,m,h}(\tau,1-k/2)$ converges in these cases.

By the theory of the resolvent kernel, the Poincaré series $P_{1/2,m,h}(\tau,s)$ has a meromorphic continuation to $\mathrm{Re}(s)>1/2$ which is analytic up to finitely many possible simple poles in the real segment $(1/2,1)$. These poles occur at points of the discrete spectrum of $\Delta_{1/2}$, so the corresponding residues are square-integrable with respect to the Petersson inner product. We define the function

$$P^{\text{sesqui}}_{1/2,m,h}(\tau) = \mathrm{CT}_{s=3/4}\left(P_{1/2,m,h}(\tau,s)\right)$$

as the constant term at $s = 3/4$ in the Laurent expansion of the meromorphic continuation of $P_{1/2,m,h}(\tau, s)$, and we let

$$R_{1/2,m,h}(\tau) = \mathrm{Res}_{s=3/4}\left(P_{1/2,m,h}(\tau,s)\right)$$

denote the residue of the Poincaré series at $s = 3/4$.

Proposition 2.3.19. *The function $P^{\text{sesqui}}_{1/2,m,h}(\tau)$ is modular of weight $1/2$ for ρ_L and satisfies the differential equation*

$$\Delta_{1/2}P^{\text{sesqui}}_{1/2,m,h}(\tau) = -\frac{1}{2}R_{1/2,m,h}(\tau) \in M_{1/2,\rho_L}. \tag{2.3.14}$$

It has the Fourier expansion

$$\begin{aligned} P^{\text{sesqui}}_{1/2,m,h}(\tau) = {} & \mathcal{M}_{m,1/2}(v)e(mu)(\mathfrak{e}_h + \mathfrak{e}_{-h}) \\ & + \sum_{h' \in L'/L}\sum_{n\in\mathbb{Q}} \mathrm{CT}_{s=3/4}\left(b_{1/2,m,h}(n,h',s)\right)\mathcal{W}_{n,1/2}(v,3/4)e(nu)\mathfrak{e}_{h'} \\ & + \sum_{h' \in L'/L}\sum_{n\geq 0} \mathrm{Res}_{s=3/4}\left(b_{1/2,m,h}(n,h',s)\right)\left(\tfrac{d}{ds}\mathcal{W}_{n,1/2}(v,s)\right)|_{s=3/4}e(nu)\mathfrak{e}_{h'}. \end{aligned}$$

In particular, it is a sesquiharmonic Maass form in the sense of [BDR13].

Proof. The Laplace equation (2.3.13) shows that $R_{1/2,m,h}(\tau)$ is harmonic and that the function $P^{\text{sesqui}}_{1/2,m,h}(\tau)$ satisfies the differential equation (2.3.14). The fact that $R_{1/2,m,h}(\tau)$ is also square-integrable implies that it is actually a holomorphic modular form of weight $1/2$ for ρ_L. This also implies that the Fourier coefficients of negative index of $P_{1/2,m,h}(\tau, s)$ are holomorphic at $s = 3/4$. Now we obtain the stated expansion from Proposition 2.3.18. □

We compute the action of $\xi_{1/2}$ on the non-harmonic part in the third line of the expansion above.

Lemma 2.3.20. *For $n \geq 0$ we have*

$$\xi_{1/2}\left(\left(\frac{d}{ds}\mathcal{W}_{n,1/2}(v,s)\right)\Big|_{s=3/4} e(nu)\right) = \begin{cases} -2v^{-1/2}, & n = 0, \\ -\sqrt{\pi}\,\Gamma(-1/2, 4\pi n v)e(-n\tau), & n > 0. \end{cases}$$

Further, for $n>0$ we have

$$\left(\frac{d}{ds}\mathcal{W}_{n,1/2}(v,s)\right)\Big|_{s=3/4} = O\big(n^{-1/2}|\log(nv)|e^{-2\pi nv}\big)$$

as $v\to\infty$ or $v\to 0$, where the implied constant does not depend on n.

Proof. For $n=0$ a direct calculation gives

$$\xi_{1/2}\left(\frac{d}{ds}\mathcal{W}_{0,1/2}(v,s)\right)\Big|_{s=3/4} = 2iv^{1/2}\frac{\partial}{\partial\tau}\left(-2\log(v)-4\Gamma'(1)+4\log(2)\right) = -2v^{-1/2}.$$

For $n>0$ we first compute the derivative in τ,

$$\begin{aligned}\xi_{1/2}\left(\mathcal{W}_{n,1/2}(v,s)e(nu)\right) &= 2iv^{1/2}e(-n\bar\tau)\frac{\partial}{\partial\tau}\left(\mathcal{W}_{n,1/2}(v,s)e^{2\pi nv}\right)\\ &= \frac{n^{-3/4}v^{1/2}}{\Gamma(s+1/4)}e(-n\bar\tau)\frac{\partial}{\partial v}\left((4\pi v)^{-1/4}e^{2\pi nv}W_{1/4,s-1/2}(4\pi nv)\right).\end{aligned}$$

Using the integral representation

$$W_{\kappa,\mu}(v) = \frac{v^{\mu+1/2}e^{-v/2}}{\Gamma(1/2+\mu-\kappa)}\int_0^\infty e^{-vt}t^{\mu-1/2-\kappa}(t+1)^{\mu-1/2+\kappa}dt,$$

valid for $\mathrm{Re}(\mu-1/2-\kappa)>-1$ by [AS64, (13.1.33), (13.2.5)], and the recurrence relation [AS64, (13.4.30)] for the W-Whittaker functions, a short calculation yields

$$\frac{\partial}{\partial v}\left(v^{-1/4}e^{v/2}W_{1/4,s-1/2}(v)\right) = -(s-1/4)(s-3/4)v^{-5/4}e^{v/2}W_{-3/4,s-1/2}(v).$$

Now we take the derivative with respect to s and plug in $s=3/4$. Using the identity

$$W_{-3/4,1/4}(v) = e^{v/2}v^{3/4}\Gamma(-1/2,v)$$

and taking everything together, we obtain the stated formula.

The growth estimates can easily be shown using the integral representation for the W-Whittaker function given above. □

Theorem 2.3.21. *Let $f\in H_{1/2,\rho_L}$ with a Fourier expansion as in (2.3.12). Then we have*

$$f-\frac{1}{2}\sum_{h\in L'/L}\sum_{m\in\mathbb{Q}}a_f(m,h)P^{\mathrm{sesqui}}_{1/2,m,h}\in M_{1/2,\rho_L}, \tag{2.3.15}$$

i.e., every harmonic Maass form in $H_{1/2,\rho_L}$ can be uniquely written as a linear combination of the sesquiharmonic Poincaré series $P^{\mathrm{sesqui}}_{1/2,m,h}(\tau)$ and a holomorphic modular *form.*

Proof. Let g be the difference in (2.3.15). Applying $\xi_{1/2}$ to the Fourier expansion of $P^{\text{sesqui}}_{1/2,m,h}$, and taking into account Lemma 2.3.20, we see that $\widetilde{g} = \xi_{1/2} g$ is a harmonic Maass form of weight 3/2 for ρ_L^* whose principal part is given by some constant in $\mathbb{C}[L'/L]$ times $v^{-1/2}$, and whose coefficients $b_{\widetilde{g}}(0,h')$ vanish for all $h' \in L'/L$. Further, $\xi_{3/2}\widetilde{g} = -\Delta_{1/2} g$ is a linear combination of the residues $R_{1/2,m,h}$, and hence a holomorphic modular form. By (2.3.7) the Petersson norm of $\xi_{3/2}\widetilde{g}$ can be evaluated as

$$(\xi_{3/2}\widetilde{g}, \xi_{3/2}\widetilde{g}) = \sum_{h' \in L'/L} \left(\frac{2}{\pi} b_{\widetilde{g}}(0,h') b_{\xi_{3/2}\widetilde{g}}(0,h') + \sum_{n<0} |n|^{-1/2} a_{\widetilde{g}}(n,h') b_{\xi_{3/2}\widetilde{g}}(-n,h') \right).$$

Note that the Petersson norm of a holomorphic modular form of weight 1/2 converges even if the form is non-cuspidal. The right-hand side vanishes by what we have said above. This implies $\xi_{3/2}\widetilde{g} = 0$, so $\widetilde{g}$ is holomorphic and g is actually a harmonic Maass form. From the Fourier expansion of g we see that $\xi_{1/2} g$ is a cusp form. Since the principal part of g vanishes by construction, we obtain $(\xi_{1/2} g, \xi_{1/2} g) = 0$ by the same argument as above. Thus $\xi_{1/2} g = 0$, which means that g is holomorphic. □

2.3.8 Estimates for the Fourier coefficients of harmonic Maass forms

As an application we show that the $\mathcal{W}$-Whittaker coefficients $b_f(n,h)$ of positive index $n > 0$ of a harmonic Maass form $f \in H_{1/2,\rho_L}$ are of polynomial growth if the $\mathcal{M}$-Whittaker coefficients $a_f(n,h)$ of negative index $n < 0$ vanish.

Theorem 2.3.22. *Let $f \in H_{1/2,\rho_L}$ be a harmonic Maass form of weight 1/2 for ρ_L.*

(a) We have

$$b_f(n,h) = O\left(e^{C\sqrt{|n|}}\right) \qquad \text{as } n \to \pm\infty,$$

for some constant $C > 0$ which is independent of n.

(b) If $a_f(n,h) = 0$ for all $n \geq 0$ and $h \in L'/L$, then

$$b_f(n,h) = O\big(|n|^{3/4}\big) \quad \text{as } n \to -\infty.$$

(c) If $a_f(n,h) = 0$ for all $n < 0$ and $h \in L'/L$, then

$$b_f(n,h) = O\big(n^{3/2}\big) \quad \text{as } n \to \infty.$$

Remark 2.3.23. 1. The estimate $O(n^{3/2})$ in part (c) is probably not optimal, but much better than the exponential growth obtain from the estimate in (a). The proof of (c) presented below is very complicated compared to the proofs of (a) and (b), so it would be desirable to find a simpler proof of (c).

2. In terms of the normalization (2.3.1) of the Fourier expansion of a harmonic Maass form, the result says that the coefficients $c_f^+(n,h)$ of the holomorphic part of f grow polynomially in n if the holomorphic principal part P_f^+ vanishes.

Proof of Theorem 2.3.22. The first statement is Lemma 3.4 in [BF04], and can be proven by the same arguments as in the proof of the usual Hecke bound for holomorphic modular forms. The second claim follows from the Hecke bound applied to the cusp form $\xi_{1/2}f$.

We now prove the third bound. Theorem 2.3.21 implies that the non-harmonic parts of the functions $P^{\text{sesqui}}_{1/2,m,h}$ cancel out in the linear combination (2.3.15). This, together with the assumption $a_f(n,h') = 0$ for $n < 0$ and $h' \in L'/L$, implies that $b_f(n,h')$ can be written as a linear combination of the constant terms $\mathrm{CT}_{s=3/4}(b_{1/2,m,h}(n,h',s))$ with $m \geq 0$, plus the coefficient $c_g(n,h')$ of a holomorphic modular form

$$g(\tau) = \sum_{h' \in L'/L} \sum_{n \geq 0} c_g(n,h') n^{-1/2} e(n\tau)\mathfrak{e}_{h'} \in M_{1/2,\rho_L}.$$

Since the latter space of holomorphic modular forms is either trivial or spanned by unary theta series, we have $c_g(n,h') = O(n^{1/2})$ as $n \to \infty$. We are left to estimate the coefficients of the Poincaré series. The basic idea is that for $m > 0$ the decay of the J-Bessel function appearing in these coefficients results in polynomial growth, whereas the growth of the I-Bessel function would yield exponential growth. The proof is quite technical due to the poor convergence of the series defining the Fourier coefficients.

If $m = 0$, then the Fourier expansion of $P^{\text{sesqui}}_{1/2,0,h}$ given in Proposition 2.3.19 and the growth of the appearing Whittaker functions (see also Lemma 2.3.20) show that

$$P^{\text{sesqui}}_{1/2,0,h}(\tau) = O(v^{1/2}), \quad \text{as } v \to \infty,$$

uniformly in u. Using the modularity of the Poincaré series, this implies

$$P^{\text{sesqui}}_{1/2,0,h}(\tau) = O(v^{-1}), \quad \text{as } v \to 0,$$

uniformly in u. For $n > 0$ the n-th Fourier coefficient of the Poincaré series is given by

$$\begin{aligned}\int_0^1 \langle P^{\text{sesqui}}_{1/2,0,h}(\tau), e(nu)\mathfrak{e}_{h'}\rangle du = {} & \mathrm{CT}_{s=3/4}\left(b_{1/2,0,h}(n,h',s)\right) n^{-1/2} e^{-2\pi n v} \\ & + \mathrm{Res}_{s=3/4}\left(b_{1/2,0,h}(n,h',s)\right)\left(\tfrac{d}{ds}\mathcal{W}_{n,1/2}(v,s)\right)\big|_{s=3/4}.\end{aligned}$$

For $0 < v < 1$ the Fourier integral can be estimated by $Cv^{-1}e^{2\pi n v}$ for some constant C which is independent of n and v. Further, we have $\mathrm{Res}_{s=3/4}\left(b_{1/2,0,h}(n,h',s)\right) = O(n^{1/2})$

as $n \to \infty$, since the residues appear as the coefficients of the holomorphic modular form $\Delta_{1/2} P^{\text{sesqui}}_{1/2,0,h}$. Finally, plugging in $v = 1/n$ and using the estimate for the derivative of the $\mathcal{W}$-Whittaker function from Lemma 2.3.20, we obtain

$$\mathrm{CT}_{s=3/4}\left(b_{1/2,0,h}(n,h',s)\right) = O(n^{3/2})$$

as $n \to \infty$.

Now let $m > 0$ and $n > 0$. By splitting the sum over c in the coefficient $b_{1/2,m,h}(n,h',s)$ at n and using that

$$J_{2s-1}(y) = \left(\frac{y}{2}\right)^{2s-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(2s+k)} \left(\frac{y}{2}\right)^{2k} \quad \text{and} \quad J_{1/2}(y) = \sqrt{\frac{2}{\pi y}}\sin(y)$$

for $y > 0$ (see [EMOT54, 8.440, 8.464]) it is easy to see that

$$\begin{aligned} &\mathrm{CT}_{s=3/4}\left(b_{1/2,m,h}(n,h',s)\right) \\ &\quad = \mathrm{CT}_{s=3/4}\left(\frac{(2\pi)^{2s}(mn)^{s-1/4}}{\Gamma(2s)} \sum_{c\neq 0} H_c(h,m,h',n)|c|^{1-2s}\right) + O(n^{3/2}). \end{aligned} \tag{2.3.16}$$

The function

$$Z(h,m,h',n;s) = \sum_{c\neq 0} H_c(h,m,h',n)|c|^{1-2s}$$

appearing above is called a Kloosterman zeta function. Estimates for (scalar valued analogs of) these zeta functions with respect to m and n have been treated by Goldfeld and Sarnak [GS83], Hejhal [Hej83] and Pribitkin [Pri00]. We sketch the arguments of [Pri00] that lead to an estimate of $Z(h,m,h',n;s)$ as $n \to \infty$. The main tool is Selberg's Poincaré series

$$\mathcal{P}_{m,h}(\tau,s) = \frac{1}{2} \sum_{(M,\phi)\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}} [v^s e(m\tau)\mathfrak{e}_h]|_{1/2,\rho_L}(M,\phi),$$

which converges absolutely and locally uniformly for $\mathrm{Re}(s) > 3/4$, and is modular of weight $1/2$ for ρ_L. Using the spectral theory of automorphic forms, Selberg [Sel65] derived the meromorphic continuation of the Poincaré series to all of $\mathbb{C}$. The meromorphic continuation of the Kloosterman zeta functions then follows from the fact that they appear in the Fourier expansion of $\mathcal{P}_{m,h}(\tau,s)$. The Fourier expansion in the vector valued setting can be found in [vPSV17], Proposition 3.1.

For $n > m > 0$, a calculation analogous to the one in [Pri00], Lemma 1, gives the equation

$$Z(h,m,h',n;1/4+s) \tag{2.3.17}$$
$$= 2^{s/2+7/2}\pi n^2 \frac{\Gamma(s+1/2)\Gamma(s+2)}{\Gamma(2s+3/2)}\Big(\big(\mathcal{P}_{m,h}(\tau,s),\mathcal{P}_{n,h'}(\tau,\bar{s}+2)\big)+R_{m,n}(s)\Big)$$

for $\mathrm{Re}(s) > 3/4$, where

$$R_{m,n}(s) = 2\sqrt{i}\sum_{c\neq 0}\frac{H_c(h,m,h',n)}{|c|^{2s-1/2}}$$
$$\times \int_{v=0}^{\infty}\int_{u=-\infty}^{\infty}\left(\sum_{p=1}^{\infty}\frac{(-\frac{m}{c^2v(u+i)})^p}{p!}\right)\frac{ve^{-2\pi nv}e^{-2\pi invu}}{(u+i)^{s+1/2}(u-i)^s}du\,dv.$$

The function $R_{m,n}(s)$ is holomorphic for $\mathrm{Re}(s) > 1/4$, so (2.3.17) extends to this domain by analytic continuation. Further, using the fact that the Kloosterman sums $H_c(h,m,h',n)$ are universally bounded together with the basic estimate $\big|\sum_{p=1}^{\infty}\frac{z^p}{p!}\big| \leq |z|$ for $\mathrm{Re}(z) < 0$, we easily obtain $R_{m,n}(1/2) = O(n^{-1})$ as $n \to \infty$.

The Poincaré series $\mathcal{P}_{n,h'}(\tau,\bar{s}+2)$ converges at $s = 1/2$, and the meromorphic continuation of $\mathcal{P}_{m,h}(\tau,s)$ has at most a simple pole at $s = 1/2$. Hence, in order to bound the constant term of $Z(h,m,h',n;s)$ at $s = 3/4$, we need to estimate the inner products

$$\big(\mathrm{CT}_{s=1/2}\big(\mathcal{P}_{m,h}(\tau,s)\big),\mathcal{P}_{n,h'}(\tau,5/2)\big)$$

and

$$\big(\mathrm{Res}_{s=1/2}\big(\mathcal{P}_{m,h}(\tau,s)\big),\big(\tfrac{d}{ds}\mathcal{P}_{n,h'}(\tau,s)\big)\big|_{s=5/2}\big).$$

This can be done using the Cauchy-Schwarz inequality and the fact that the norms $\|\mathcal{P}_{n,h'}(\tau,5/2)\|$ and $\|\big(\frac{d}{ds}\mathcal{P}_{n,h'}(\tau,s)\big)|_{s=5/2}\|$ are $O(n^{-7/4+\varepsilon})$ as $n \to \infty$, compare [Pri00], Lemma 3. Taking everything together, we finally obtain the desired bound

$$\mathrm{CT}_{s=3/4}\big(b_{1/2,m,h}(n,h',s)\big) = O\big(n^{3/2}\big)$$

as $n \to \infty$. The proof is finished. □

2.4 Theta functions

In this section we introduce the theta functions that we will employ as kernel functions for the lifts we investigate in this work. As before we let $L \subseteq V$ be an even lattice with dual lattice L' and we let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which maps L to itself and acts trivially on L'/L.

2.4.1 The Siegel, Kudla-Millson, Millson and Shintani theta functions

For $z = x + iy \in \mathbb{H}$ we let

$$g_z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

be a matrix with $g_z i = z$. Then the vectors

$$\begin{aligned} X_1(z) &= \frac{1}{\sqrt{2Ny}} \begin{pmatrix} -x & x^2+y^2 \\ -1 & x \end{pmatrix} = g_z e_1, \\ X_2(z) &= \frac{1}{\sqrt{2Ny}} \begin{pmatrix} x & -x^2+y^2 \\ 1 & -x \end{pmatrix} = g_z e_2, \\ X_3(z) &= \frac{1}{\sqrt{2Ny}} \begin{pmatrix} y & -2xy \\ 0 & -y \end{pmatrix} = g_z e_3, \end{aligned}$$

form an orthogonal basis of $V(\mathbb{R})$ with

$$(X_1(z), X_1(z)) = 1 \quad \text{and} \quad (X_2(z), X_2(z)) = (X_3(z), X_3(z)) = -1.$$

In particular, for each $z \in \mathbb{H}$ we have a corresponding isometry

$$v_z : V(\mathbb{R}) \to \mathbb{R}^{1,2}, \qquad v_z\left(\sum_{i=1}^{3} \alpha_i X_i(z)\right) = (\alpha_1, \alpha_2, \alpha_3).$$

For $z = x + iy \in \mathbb{H}$ and $X = \left(\begin{smallmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{smallmatrix}\right) \in V(\mathbb{R})$ we define the quantities

$$\begin{aligned} Q_X(z) &= \sqrt{2N}y(X, X_2(z) + iX_3(z)) = N(x_3 z^2 - 2x_2 z - x_1), \\ p_X(z) &= \sqrt{2}(X, X_1(z)) = -\frac{\sqrt{N}}{y}(x_3|z|^2 - 2x_2 x - x_1). \end{aligned}$$

Under the isometry v_z they correspond to the polynomials $\sqrt{2}\alpha_1$ and $-\sqrt{2N}y(\alpha_2 + i\alpha_3)$ on $\mathbb{R}^{1,2}$, which are harmonic and homogeneous of degree $(1, 0)$ and $(0, 1)$, respectively. For $M \in \mathrm{SL}_2(\mathbb{R})$ we have the transformation rules

$$\begin{aligned} Q_X(Mz) &= j(M, z)^{-2} Q_{M^{-1}X}(z), \\ p_X(Mz) &= p_{M^{-1}X}(z), \end{aligned} \tag{2.4.1}$$

and the useful identities

$$Q(X_z) = \frac{1}{4}p_X^2(z),$$
$$Q(X_{z^\perp}) = -\frac{1}{4Ny^2}|Q_X(z)|^2,$$

which can be verified by a direct calculation. Here X_z and $X_{z^\perp}$ denote the orthogonal projections of X to the positive line $\mathbb{R}X_1(z)$ and the negative plane $\mathbb{R}X_2(z) + \mathbb{R}X_3(z)$, respectively. It is common in the literature on theta functions and theta lifts in signature $(1, 2)$ to define the quantity

$$R(X, z) = -2Q(X_{z^\perp}) = \frac{1}{2}p_X^2(z) - (X, X),$$

which is non-negative and equals 0 if and only if $X \in \mathbb{R}X_1(z)$. Then we have

$$2\pi i(\tau Q(X_z) + \bar{\tau}Q(X_{z^\perp})) = -2\pi vR(X, z) + 2\pi i\tau Q(X).$$

Remark 2.4.1. For the lattice L from Section 2.2.5 and $X = \left(\begin{smallmatrix} -b/2N & -c/N \\ a & b/2N \end{smallmatrix}\right) \in L'$ we have

$$Q_X(z) = aNz^2 + bz + c$$

and

$$p_X(z) = -\frac{aN|z|^2 + bx + c}{y\sqrt{N}}.$$

In particular, these quantities are related to CM points and geodesics associated to the quadratic form $Q_X = [aN, b, c]$ corresponding to X as in Section 2.2.5. If $Q(X) > 0$ then $Q_X(z) = 0$ if and only if $z = z_X$ is the CM point associated to X, and if $Q(X) < 0$ then $p_X(z) = 0$ if and only if z lies on the geodesic c_X.

For $k \in \mathbb{Z}_{\geq 0}$ we now define the theta functions

$$\Theta(\tau, z) = v\Theta(\tau, v_z, 1),$$
$$\Theta_{KM}(\tau, z) = v\Theta(\tau, v_z, p_X^2(z)),$$
$$\Theta_{M,k}(\tau, z) = v^{1+k}\Theta\left(\tau, v_z, p_X(z)Q_X^k(\bar{z})\right),$$
$$\Theta_{Sh,k}(\tau, z) = v^{1/2}\Theta\left(\tau, v_z, y^{-2k-2}Q_X^{k+1}(\bar{z})\right),$$

which we call the **Siegel**, the **Kudla-Millson**, the **Millson** and the **Shintani theta function**, respectively. Here $\Theta(\tau, v, p)$ denotes the theta function associated to an isometry $v : V(\mathbb{R}) \to \mathbb{R}^{1,2}$ and a polynomial p on $\mathbb{R}^{1,2}$ as in Section 2.1.3. Note that we wrote $p_X^2(z)$ instead of $2\alpha_1^2$, and $Q_X(\bar{z})$ instead of $\overline{Q_X(z)}$ by an inessential abuse of notation.

Remark 2.4.2. The above theta functions have been studied in many recent works, for example [BF04, BF06, Höv12, AE13, BFI15, AGOR15, Cra15, Alf15], but they date back much earlier. For example, the Siegel theta function was considered by Siegel [Sie51], and the Shintani theta function was used by Shintani [Shi75], Niwa [Niw74], Cipra [Cip83], and Gross, Kohnen and Zagier [ZK81, GKZ87] (they actually used a holomorphic version of the Shintani kernel), to study the Shimura-Shintani correspondence between cusp forms of integral weight $2k+2$ and half-integral weight $3/2+k$.

In their works [KM86] and [KM90], Kudla and Millson defined certain Schwartz functions φ and ψ with values in the differential forms of degree q and $q-1$ on a rational quadratic space V of signature (p,q), and they used the associated theta functions and lifts to show that the generating series of cycle integrals of compactly supported differential forms along certain special cycles are cusp forms. The 2-form $\Theta_{KM}(\tau,z)d\mu(z)$ is the theta function associated to the Schwartz form form φ in the case of signature $(1,2)$, and $\Theta_{M,0}(\tau,z)$ is the theta function associated to the Schwartz function ψ in signature $(2,1)$. In fact, the 1-form $\Theta_{Sh,0}(\tau,z)dz+\overline{\Theta_{Sh,0}(\tau,z)}d\bar{z}$ is also an instance of a theta function associated to φ, now in signature $(2,1)$. The Schwartz forms φ and ψ by construction satisfy a certain differential equation which translates into a differential equation for the Shintani and the Millson theta functions (see Lemma 2.4.9), and which is of great importance for our work.

Remark 2.4.3. Let us give more explicit formulas for the theta functions. Since the polynomials corresponding to $p_X(z)Q_X^k(\bar{z})$ and $y^{-2k-2}Q_X^{k+1}(\bar{z})$ are harmonic for each $k\geq 0$, the Millson and the Shintani theta functions can be explicitly written as

$$\Theta_{M,k}(\tau,z)=v^{1+k}\sum_{h\in L'/L}\sum_{X\in h+L}p_X(z)Q_X^k(\bar{z})e(\tau Q(X_z)+\bar{\tau}Q(X_{z^\perp}))\mathfrak{e}_h$$

and

$$\Theta_{Sh,k}(\tau,z)=v^{1/2}\sum_{h\in L'/L}\sum_{X\in h+L}y^{-2k-2}Q_X^{k+1}(\bar{z})e(\tau Q(X_z)+\bar{\tau}Q(X_{z^\perp}))\mathfrak{e}_h.$$

On the other hand, the function $p_X^2(z)$ corresponds to the polynomial $2\alpha_1^2$, and the polynomial

$$\exp\left(-\frac{\Delta}{8\pi\operatorname{Im}(\tau)}\right)2\alpha_1^2=2\alpha_1^2-\frac{1}{2\pi\operatorname{Im}(\tau)}$$

corresponds to the function $p_X^2(z)-\frac{1}{2\pi v}$, so the Kudla-Millson theta function equals

$$\Theta_{KM}(\tau,z)=\sum_{h\in L'/L}\sum_{X\in h+L}\left(vp_X^2(z)-\frac{1}{2\pi}\right)e(\tau Q(X_z)+\bar{\tau}Q(X_{z^\perp}))\mathfrak{e}_h,$$

which is the formula often used in the literature, see [BF06], Section 3, for example.

Next, we summarize the transformation properties of the theta functions.

Proposition 2.4.4. *Let $k \in \mathbb{Z}$, $k \geq 0$.*

1. *The Siegel theta function $\Theta(\tau, z)$ has weight $-1/2$ for ρ_L in τ and weight 0 for Γ in z.*

2. *The Kudla-Millson theta function $\Theta_{KM}(\tau, z)$ has weight $3/2$ for ρ_L in τ and weight 0 for Γ in z.*

3. *The Millson theta function $\Theta_{M,k}(\tau, z)$ has weight $1/2-k$ for ρ_L in τ and $\overline{\Theta_{M,k}(\tau, z)}$ has weight $-2k$ in for Γ in z.*

4. *The Shintani theta function $\overline{\Theta_{Sh,k}(\tau, z)}$ has weight $k + 3/2$ for ρ_L^* in τ and $\Theta_{Sh,k}(\tau, z)$ has weight $2k + 2$ for Γ in z.*

Proof. The behaviour in z follows from the rules (2.4.1), and the behaviour in τ follows from Theorem 2.1.1 if we use that $p_X(z)$ and $Q_X(z)$ are homogenous of degree $(1, 0)$ and $(0, 1)$, respectively. ☐

2.4.2 Growth of the theta functions at the cusps

Next, we want to investigate the growth of the above theta functions at the cusps of Γ. To describe this in a convenient way, we follow the ideas of [BFI15, Section 2.2] and define certain unary theta functions associated to the cusps.

For an isotropic line $\ell \in \mathrm{Iso}(V)$ the space $W_\ell = \ell^\perp/\ell$ is a unary negative definite quadratic space with the quadratic form $Q(X + \ell) := Q(X)$, and

$$K_\ell = (L \cap \ell^\perp)/(L \cap \ell)$$

is an even lattice with dual lattice

$$K_\ell' = (L' \cap \ell^\perp)/(L' \cap \ell).$$

The vector $X_\ell = \sigma_\ell.X_3(i)$ is a basis of W_ℓ with $(X_\ell, X_\ell) = -1$, and for $k \in \mathbb{Z}_{\geq 0}$ the polynomial $p_{\ell,k}(X) = (-\sqrt{2N}i(X, X_\ell))^k$ is homogeneous of degree $(0, k)$. We let $\Theta_{\ell,k}(\tau)$ be the theta function associated to K_ℓ and $p_{\ell,k}$. By Theorem 2.1.1 the complex conjugate $\overline{\Theta_{\ell,k}(\tau)}$ is an almost holomorphic modular form of weight $k + 1/2$ for the dual Weil representation of K_ℓ. Using [Bru02, Lemma 5.6.], it gives rise to an almost holomorphic modular form of weight $k + 1/2$ for the dual Weil representation ρ_L^* of L, which we also denote by $\overline{\Theta_{\ell,k}(\tau)}$. For $k = 0$ it is a holomorphic modular form, and for $k = 1$ it is a cusp form.

Remark 2.4.5. We want to make this construction more explicit. Let us write $\overline{\Theta_{\ell,k}(\tau)} = \sum_{h\in L'/L} \sum_{m\geq 0} b_{\ell,k}(m, h, v)e(m\tau)\mathfrak{e}_h$ for the Fourier expansion of the theta function. For

$m = 0$ we have $b_{\ell,k}(0,h,v) = 0$ unless $\ell \cap (L+h) \neq \emptyset$, in which case we have

$$b_{\ell,k}(0,h,v) = \frac{(-\sqrt{N}i)^k}{(4\pi v)^{k/2}} H_k\left(0\right),$$

where $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$ is the k-th Hermite polynomial. Further, for $m > 0$ we have $b_{\ell,k}(m,h,v) = 0$ unless m/N is a square and there exists a vector $X \in L_{-m,h}$ orthogonal to ℓ, in which case it equals

$$b_{\ell,k}(m,h,v) = (\pm 1)^k \frac{(-\sqrt{N}i)^k}{(4\pi v)^{k/2}} H_k\left(2\sqrt{\pi m v}\right)$$

if $h \neq -h \bmod L$, and $1+(-1)^k$ times this expression if $h = -h \bmod L$. If $h \neq -h \bmod L$ the sign is $+1$ if $\ell = \ell_X$, and -1 if $\ell = \ell_{-X}$ (if $h = -h \bmod L$ then the coefficient vanishes if k is odd, and the sign does not matter if k if even).

From the Fourier expansions we find that the theta functions are related by

$$\xi_{3/2+k}\overline{\Theta_{\ell,k+1}(\tau)} = -\frac{Nk(k+1)}{4\pi} v^{k-1/2}\Theta_{\ell,k-1}(\tau),$$

for $k > 0$.

All of these formulas can be proven by straightforward, but tedious computations. Since we will not use them, we omit the proofs.

Remark 2.4.6. If L is the lattice from Section 2.2.5, then the theta functions $\overline{\Theta_{\ell,k}(\tau,z)}$ for $k = 0,1$ and $\ell = \infty$ agree (up to a simple constant) with the unary theta functions $\theta_{1/2,N}(\tau)$ and $\theta_{3/2,N}(\tau)$ defined in Section 2.3.6. More generally, for other cusps ℓ, the functions $\overline{\Theta_{\ell,k}(\tau,z)}$ are obtained from $\theta_{1/2,N}(\tau)$ and $\theta_{3/2,N}(\tau)$ by applications of suitable Atkin-Lehner and level raising operators to $\theta_{1/2,N}(\tau)$ and $\theta_{3/2,N}(\tau)$.

We can now estimate the growth of our theta functions at the cusps of Γ.

Proposition 2.4.7. *Let $\ell \in \Gamma\backslash \operatorname{Iso}(V)$ be a cusp of Γ.*

1. *For the Siegel theta function we have*

$$\Theta(\tau,\sigma_\ell z) = y\frac{1}{\sqrt{N}\beta_\ell} v^{1/2}\Theta_{\ell,0}(\tau) + O(e^{-Cy^2}),$$

as $y \to \infty$, uniformly in x, for some constant $C > 0$.

2. *For the Kudla-Millson theta function we have*

$$\Theta_{KM}(\tau,\sigma_\ell z) = O(e^{-Cy^2}),$$

as $y \to \infty$, uniformly in x, for some constant $C > 0$.

3. *For the Millson theta function we have*

$$\Theta_{M,k}(\tau, \sigma_\ell z) = O(e^{-Cy^2}),$$

if $k = 0$, and

$$j(\sigma_\ell, \bar{z})^{2k}\Theta_{M,k}(\tau, \sigma_\ell z) = -y^{k+1}\frac{k}{2\pi\beta_\ell}v^{k-1/2}\Theta_{\ell,k-1}(\tau) + O(e^{-Cy^2}),$$

if $k > 0$, as $y \to \infty$, uniformly in x, for some constant $C > 0$.

4. *For the Shintani theta function we have*

$$j(\sigma_\ell, z)^{-2k-2}\Theta_{Sh,k}(\tau, \sigma_\ell z) = y^{-k}\frac{1}{\sqrt{N}\beta_\ell}\Theta_{\ell,k+1}(\tau) + O(e^{-Cy^2}),$$

as $y \to \infty$, uniformly in x, for some constant $C > 0$.

Moreover, all of the partial derivates of the functions hidden in the O-notation are square exponentially decreasing as $y \to \infty$.

Proof. Using the rules (2.4.1) we can write

$$\begin{aligned} &j(\sigma_\ell, \bar{z})^{2k}\Theta_{M,k}(\tau, \sigma_\ell z) \\ &\quad= v^{1+k}\sum_{h\in(\sigma_\ell^{-1}L)'/(\sigma_\ell^{-1}L)}\ \sum_{X\in h+(\sigma_\ell^{-1}L)} p_X(z)Q_X^k(\bar{z})e(\tau Q(X_z) + \bar{\tau}Q(X_{z^\perp}))\mathfrak{e}_h, \end{aligned}$$

and similarly for the other theta functions, so we can equivalently estimate the growth of the theta functions for the lattice $\sigma_\ell^{-1}L$ at the cusp ∞. The result now follows from Theorem 5.2 in [Bor98] applied to the lattice $\sigma_\ell^{-1}L$ and the primitive isotropic vector $\left(\begin{smallmatrix} 0 & \beta_\ell \\ 0 & 0 \end{smallmatrix}\right) \in \ell_\infty \cap \sigma_\ell^{-1}L$. □

2.4.3 Differential equations for theta functions

The theta functions we just defined satisfy some interesting differential equations, some of which we collect now. All of the following identities can be checked by a direct computation using the rules

$$\begin{aligned} &\frac{\partial}{\partial z}y^{-2}Q_X(z) = -i\sqrt{N}y^{-2}p_X(z), &&\frac{\partial}{\partial z}p_X(z) = -\frac{i}{2\sqrt{N}}y^{-2}Q_X(\bar{z}), \\ &\frac{\partial}{\partial z}R(X, z) = -\frac{i}{2\sqrt{N}}y^{-2}p_X(z)Q_X(\bar{z}), &&y^{-2}Q_X(z)Q_X(\bar{z}) = 2NR(X, z). \end{aligned} \tag{2.4.2}$$

Since the computations are rather lengthy and not very enlightening, we skip the proofs.

Lemma 2.4.8. *For $k \geq 0$, we have*

$$\begin{aligned}
\Delta_{1/2,\tau} v^{-1/2}\overline{\Theta(\tau,z)} &= \frac{1}{4}\Delta_{0,z} v^{-1/2}\overline{\Theta(\tau,z)},\\
\Delta_{3/2,\tau}\Theta_{KM}(\tau,z) &= \frac{1}{4}\Delta_{0,z}\Theta_{KM}(\tau,z),\\
\Delta_{1/2-k,\tau}\Theta_{M,k}(\tau,z) &= \frac{1}{4}\overline{\Delta_{-2k,z}\overline{\Theta_{M,k}(\tau,z)}},\\
\overline{\Delta_{k+3/2,\tau}\overline{\Theta_{Sh,k}(\tau,z)}} &= \frac{1}{4}\Delta_{2k+2,z}\Theta_{Sh,k}(\tau,z).
\end{aligned}$$

Proof. Compare [BFI15, Lemma 5.1], [Höv12, Proposition 3.10] and [Bru02, Proposition 4.5], □

The Millson and the Shintani theta function are related by the following identities.

Lemma 2.4.9. *For $k \geq 0$ we have*

$$\xi_{1/2-k,\tau}\Theta_{M,k}(\tau,z) = \frac{1}{2\sqrt{N}}\xi_{2k+2,z}\Theta_{Sh,k}(\tau,z)$$

and

$$\xi_{3/2+k,\tau}\overline{\Theta_{Sh,k}(\tau,z)} = \frac{\sqrt{N}}{2}\xi_{-2k,z}\overline{\Theta_{M,k}(\tau,z)}.$$

Proof. Compare [BKV13, Lemma 3.3] or [Cra15, Lemma 7.2.1]. □

The Siegel and the Kudla-Millson theta functions satisfy the following differential equations.

Lemma 2.4.10. *We have*

$$R_{-1/2,\tau}\Theta(\tau,z) = -\pi\Theta_{KM}(\tau,z)$$

and

$$L_{3/2,\tau}\Theta_{KM}(\tau,z) = \frac{1}{4\pi}\Delta_{0,z}\Theta(\tau,z).$$

Proof. Compare [BFI15, Remark 5.3] and [BF04, Theorem 4.4]. □

We will also need the following relations between Millson theta functions of different weights and the Millson and Kudla-Millson theta functions:

Lemma 2.4.11. *For $k \geq 0$ we have*

$$L_{-2k-2,z}L_{-2k,z}\overline{L_{1/2-k,\tau}\Theta_{M,k}(\tau,z)} = \frac{\pi}{N}\left(\Delta_{-2k-4,z} - 4k - 6\right)\overline{\Theta_{M,k+2}(\tau,z)}.$$

Further, we have

$$L_{0,z}\overline{L_{3/2,\tau}\Theta_{KM}(\tau,z)} = -\frac{1}{2\sqrt{N}}(\Delta_{-2,z}-2)\overline{\Theta_{M,1}(\tau,z)}.$$

Proof. This can be shown by a direct calculation using the identities (2.4.2). □

2.4.4 Twisted theta functions

We now explain how to obtain twisted versions of the theta functions defined above. Throughout this section we let L be the lattice defined in Section 2.2.5 and we let $\Gamma = \Gamma_0(N)$.

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant (possibly 1) and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2(4N)$. We consider the rescaled lattice $L^{(\Delta)} = \Delta L$ together with the quadratic form

$$Q_\Delta(X) = \frac{1}{|\Delta|}Q(X).$$

The corresponding bilinear form is given by

$$(X,Y)_\Delta = \frac{1}{|\Delta|}(X,Y),$$

and the dual lattice of $L^{(\Delta)}$ is equal to L'.

Following [GKZ87] we define a generalized genus character for $\delta = \left(\begin{smallmatrix} -b/2N & -c/N \\ a & b/2N \end{smallmatrix}\right) \in L'$ by

$$\chi_\Delta(\delta) = \chi_\Delta([aN,b,c]) := \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \Delta | b^2-4Nac,\ (b^2-4Nac)/\Delta \text{ is a} \\ & \text{square mod } 4N \text{ and } \gcd(a,b,c,\Delta)=1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $Q_\delta = [aN,b,c]$ is the integral binary quadratic form corresponding to δ, and n is any integer prime to Δ represented by one of the quadratic forms $[N_1a,b,N_2c]$ with $N_1N_2 = N$ and $N_1,N_2 > 0$. Note that the function χ_Δ is invariant under the action of $\Gamma_0(N)$.

Since $\chi_\Delta(\delta)$ depends only on $\delta \in L'$ modulo ΔL, we can view it as a function on the discriminant group $L'/L^{(\Delta)}$. Let $\rho_{L^{(\Delta)}}$ be the representation corresponding to $L^{(\Delta)}$. In [AE13] it was shown that we obtain an intertwiner of the Weil representations corresponding to L and $L^{(\Delta)}$ via χ_Δ.

Proposition 2.4.12 ([AE13, Proposition 3.2.]). *Let $\pi : L'/L^{(\Delta)} \to L'/L$ be the natural projection. For $h \in L'/L$ we define*

$$\psi_{\Delta,r}(\mathfrak{e}_h) = \sum_{\substack{\delta \in L'/L^{(\Delta)} \\ \pi(\delta) = rh \\ Q_\Delta(\delta) \equiv \operatorname{sgn}(\Delta) Q(h)\, (\mathbb{Z})}} \chi_\Delta(\delta)\mathfrak{e}_\delta. \tag{2.4.3}$$

Then $\psi_{\Delta,r} : L'/L \to L'/L^{(\Delta)}$ defines an intertwining linear map between the representations $\tilde{\rho}_L$ and $\rho_{L^{(\Delta)}}$, where

$$\tilde{\rho}_L = \begin{cases} \rho_L & \text{if } \Delta > 0, \\ \rho_L^* & \text{if } \Delta < 0. \end{cases}$$

We obtain twisted theta functions by setting

$$\Theta_{\Delta,r}(\tau, z, p) = \sum_{h \in L'/L} \left\langle \psi_{\Delta,r}(\mathfrak{e}_h), \overline{\Theta^{(\Delta)}(\tau, z, p)} \right\rangle \mathfrak{e}_h,$$

where $\Theta^{(\Delta)}(\tau, z, p)$ is the theta function associated to $L^{(\Delta)}$ and a polynomial p.

It is easy to check that the twisted versions of the Siegel, Millson, Kudla-Millson and Shintani theta functions have the same transformation behaviour as their untwisted counterparts (see Proposition 2.4.4) and satisfy the same growth estimates (see Proposition 2.4.7) and differential equations if we replace ρ_L by $\tilde{\rho}_L$, N by $N/|\Delta|$ and $\Theta_{\ell,k}$ by

$$\Theta_{\ell,k,\Delta,r} = \sum_{h \in L'/L} \left\langle \psi_{\Delta,r}(\mathfrak{e}_h), \overline{\Theta_{\ell,k}^{(\Delta)}} \right\rangle \mathfrak{e}_h.$$

Example 2.4.13. The twisted Millson theta function is explicitly given by

$$\begin{aligned} &\Theta_{M,k,\Delta,r}(\tau, z) \\ &\quad = v^{1+k} \sum_{h \in L'/L} \sum_{\substack{X \in L + rh \\ Q(X) \equiv \Delta Q(h)(\Delta)}} \chi_\Delta(X) \frac{p_X(z) Q_X^k(\bar{z})}{|\Delta|^{(k+1)/2}} e(\tau Q_\Delta(X_z) + \bar{\tau} Q_\Delta(X_{z^\perp}))\mathfrak{e}_h, \end{aligned}$$

and the twisted Shintani theta function is given by

$$\begin{aligned} &\Theta_{Sh,k,\Delta,r}(\tau, z) \\ &\quad = v^{1/2} \sum_{h \in L'/L} \sum_{\substack{X \in L + rh \\ Q(X) \equiv \Delta Q(h)(\Delta)}} \chi_\Delta(X) \frac{y^{-2k-2} Q_X^{k+1}(\bar{z})}{|\Delta|^{(k+1)/2}} e(\tau Q_\Delta(X_z) + \bar{\tau} Q_\Delta(X_{z^\perp}))\mathfrak{e}_h. \end{aligned}$$

3 The Millson Theta Lift

3.1 The Millson and the Shintani theta lift

Let $k \in \mathbb{Z}_{\geq 0}$ and let $F \in H^{+}_{-2k}(\Gamma)$ be a harmonic Maass form. We would like to integrate F against the Millson theta function $\Theta_{M,k}(\tau, z)$ on $M = \Gamma\backslash\mathbb{H}$ to obtain a function that transforms like a modular form of weight $1/2-k$. Unfortunately, Proposition 2.4.7 shows that the integral does not converge for $k > 0$, so it has to be regularized in a suitable way. Using the regularization of [Bor98], we define the Millson theta lift by

$$I^M(F, \tau) = \lim_{T\to\infty} \int_{M_T} F(z)\Theta_{M,k}(\tau, z)y^{-2k}d\mu(z), \qquad \left(d\mu(z) = \frac{dx\,dy}{y^2}\right).$$

Note that we integrate in the orthogonal variable z here. The integral in the symplectic variable τ was considered previously in [Höv12], [BKV13] and [Cra15]. It was shown that the corresponding lift has jump singularities along certain geodesics in the upper-half plane, which led to the discovery of locally harmonic Maass forms. Similarly, the theta lifts investigated in the fundamental works of Borcherds [Bor98] and Bruinier [Bru02] (which are integrals in the τ-variable) have singularities along Heegner divisors in $\mathbb{H}$. In contrast to these singular lifts, it turns out that the Millson theta lift is in fact harmonic on the upper half-plane.

Proposition 3.1.1. *For $k \in \mathbb{Z}_{\geq 0}$ the Millson theta lift $I^M(F, \tau)$ of $F \in H^{+}_{-2k}(\Gamma)$ is a harmonic function that transforms like a modular form of weight $1/2 - k$ for ρ_L.*

Proof. Using the truncated fundmantal domain $\mathcal{F}_T(\Gamma)$ for M_T given in Section 2.2.2, we see that it suffices to show that the limit

$$\lim_{T\to\infty} \int_1^T \int_0^{\alpha_\ell} F_\ell(z) j(\sigma_\ell, \bar{z})^{2k} \Theta_{M,k}(\tau, \sigma_\ell z) y^{-2k-2} dx\,dy$$

exists for every cusp $\ell \in \operatorname{Iso}(V)$, where $F_\ell = F|_{-2k}\sigma_\ell$. For $k = 0$ Proposition 2.4.7 states that the Millson theta function is square exponentially decreasing at all cusps, so the integral actually converges without regularization in this case. For $k > 0$ we see by the same proposition that it suffices to show that

$$\lim_{T\to\infty} \int_1^T \int_0^{\alpha_\ell} F_\ell(z) y^{-k-1} dx\,dy$$

exists. But the integral over x picks out the constant coefficient $a_\ell^+(0)$ of F_ℓ, and the limit of the remaining integral over y gives $\frac{1}{k}$. This shows that $I^M(F,\tau)$ is well-defined. The transformation behaviour of the Millson theta function implies that $I^M(F,\tau)$ has weight $1/2-k$ for ρ_L.

To prove that $I^M(F,\tau)$ is harmonic we first use Lemma 2.4.8 to write

$$\begin{aligned}\Delta_{1/2-k,\tau} I^M(F,\tau) &= \lim_{T\to\infty}\int_{M_T} F(z)\Delta_{1/2-k,\tau}\Theta_{M,k}(\tau,z)y^{-2k}d\mu(z)\\ &= \lim_{T\to\infty}\frac{1}{4}\int_{M_T} F(z)\overline{\Delta_{-2k,z}\overline{\Theta_{M,k}(\tau,z)}}y^{-2k}d\mu(z).\end{aligned}$$

By Lemma 2.3.9, i.e. Stokes' theorem, we have

$$\begin{aligned}&\int_{M_T} F(z)\overline{\Delta_{-2k,z}\overline{\Theta_{M,k}(\tau,z)}}y^{-2k}d\mu(z) - \int_{M_T}\Delta_{-2k,z}F(z)\Theta_{M,k}(\tau,z)y^{-2k}d\mu(z)\\ &= \int_{\partial M_T} F(z)\xi_{-2k,z}\overline{\Theta_{M,k}(\tau,z)}dz - \overline{\int_{\partial M_T}\xi_{-2k,z}F(z)\overline{\Theta_{M,k}(\tau,z)}dz}.\end{aligned}$$

As pointed out in Remark 2.3.8, we can write the boundary integrals above as sums of horizontal boundary pieces corresponding to the cusps of Γ. Then it follows easily from the growth estimates in Proposition 2.4.7 that the boundary integrals vanish in the limit. Since F is harmonic, we obtain $\Delta_{1/2-k,\tau}I^M(F,\tau)=0$. □

We define the Shintani theta lift of a cusp form $G \in S_{2k+2}(\Gamma)$ by

$$I^{Sh}(G,\tau) = \int_M G(z)\overline{\Theta_{Sh,k}(\tau,z)}y^{2k+2}d\mu(z).$$

The rapid decay of G at the cusps and similar arguments as above show that $I^{Sh}(G,\tau)$ converges to a harmonic function which transforms like a modular form of weight $3/2+k$ for the dual Weil representation ρ_L^*. The Millson and the Shintani theta lifts are related by the following identity.

Proposition 3.1.2. *For $F \in H_0^+(\Gamma)$ with constant coefficients $a_\ell^+(0)$ at the cusps we have*

$$\xi_{1/2,\tau}(I^M(F,\tau)) = -\frac{1}{2\sqrt{N}}I^{Sh}(\xi_{0,z}F,\tau) + \frac{1}{2N}\sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)}\varepsilon_\ell\overline{a_\ell^+(0)\Theta_{\ell,1}(\tau)},$$

and for $k \in \mathbb{Z}_{>0}$ and $F \in H_{-2k}^+(\Gamma)$ we have

$$\xi_{1/2-k,\tau}(I^M(F,\tau)) = -\frac{1}{2\sqrt{N}}I^{Sh}(\xi_{-2k,z}F,\tau).$$

Proof. By Lemma 2.4.9 we have for $k \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}\xi_{1/2-k,\tau}(I^M(F,\tau)) &= \lim_{T\to\infty} \int_{M_T} \overline{F(z)}\xi_{1/2-k,\tau}\Theta_{M,k}(\tau,z)y^{-2k}d\mu(z) \\ &= \lim_{T\to\infty} \frac{1}{2\sqrt{N}} \int_{M_T} \overline{F(z)}\xi_{2k+2,z}\Theta_{Sh,k}(\tau,z)y^{-2k}d\mu(z).\end{aligned}$$

Using Stokes' theorem in the version given in Lemma 2.3.7 we obtain

$$\begin{aligned}\int_{M_T} \overline{F(z)}\xi_{2k+2,z}\Theta_{Sh,k}(\tau,z)y^{-2k}d\mu(z) = &- \int_{M_T} \xi_{-2k,z}F(z)\overline{\Theta_{Sh,k}(\tau,z)}y^{2k+2}d\mu(z) \\ &- \overline{\int_{\partial M_T} F(z)\Theta_{Sh,k}(\tau,z)dz}.\end{aligned}$$

The limit as $T \to \infty$ of the first line on the right-hand side is $-I^{Sh}(\xi_{-2k,z}F,\tau)$. The boundary integral can be written as

$$-\overline{\int_{\partial M_T} F(z)\Theta_{Sh,k}(\tau,z)dz} = \sum_{\ell\in\Gamma\backslash \operatorname{Iso}(V)} \overline{\int_{iT}^{\alpha_\ell+iT} F_\ell(z)j(\sigma_\ell,z)^{-2k-2}\Theta_{Sh,k}(\tau,\sigma_\ell z)dz},$$

where $F_\ell = F|_{-2k}\sigma_\ell$. Using Proposition 2.4.7 and carrying out the integral we see that the right-hand side vanishes in the limit if $k > 0$ and equals

$$\frac{1}{\sqrt{N}} \sum_{\ell\in\Gamma\backslash \operatorname{Iso}(V)} \varepsilon_\ell \overline{a_\ell^+(0)\Theta_{\ell,1}(\tau)}$$

if $k = 0$. This completes the proof. $\square$

We summarize the most important mapping properties of the Millson and the Shintani theta lift in the following theorem.

Theorem 3.1.3.

1. *The Millson theta lift maps $H^+_{-2k}(\Gamma)$ to $H^+_{1/2-k,\rho_L}$ for $k \geq 0$.*
2. *The Millson theta lift maps $M^!_0(\Gamma)$ to $H^+_{1/2,\rho_L}$ and $M^!_{-2k}(\Gamma)$ to $M^!_{1/2-k,\rho_L}$ for $k > 0$.*
3. *The Shintani theta lift maps $S_{2k+2}(\Gamma)$ to $S_{3/2+k,\rho^*_L}$ for $k \geq 0$.*

Proof. For the first item it remains to compute the Fourier expansion of the Millson theta lift, which will be done in Section 3.3. The second claim then follows immediatly from Proposition 3.1.2 if we use that ξ_{-2k} annihilates holomorphic functions and that $\overline{\Theta_{\ell,1}(\tau)}$ is a cusp form of weight $3/2$ for ρ^*_L. The third item then follows by combining the first item with Proposition 3.1.2 and the fact that $\xi_{-2k} : H^+_{-2k}(\Gamma) \to S_{2k+2}(\Gamma)$ is surjective. $\square$

3.2 Higher weight theta lifts via differential operators

The square exponential decay of the $k = 0$ Millson theta function and the Kudla-Millson theta function at the cusps implies that the integral of a harmonic Maass form $F \in H_0^+(\Gamma)$ against each of the two theta functions over $M = \Gamma\backslash\mathbb{H}$ converges without regularization. Following ideas of [BO13] and [Alf15], we define theta lifts of $F \in H_{-2k}^+(\Gamma)$ by first raising it to a Γ-invariant function, integrating it against the two theta functions, and then applying suitable differential operators to make the result harmonic again. To make this precise let $k \in \mathbb{Z}_{\geq 0}$ and $F \in H_{-2k}^+(\Gamma)$. We define

$$\Lambda^M(F,\tau) = \begin{cases} L_{1/2,\tau}^{k/2} \displaystyle\int_{\Gamma\backslash\mathbb{H}} R_{-2k,z}^k F(z)\Theta_{M,0}(\tau,z)d\mu(z), & \text{if } k \text{ is even,} \\ L_{3/2,\tau}^{(k+1)/2} \displaystyle\int_{\Gamma\backslash\mathbb{H}} R_{-2k,z}^k F(z)\Theta_{KM}(\tau,z)d\mu(z), & \text{if } k \text{ is odd.} \end{cases}$$

This lift has been considered by Alfes-Neumann in her thesis [Alf15], where it was called the Bruinier-Funke lift.

Proposition 3.2.1. *Let $k \in \mathbb{Z}_{\geq 0}$ and $F \in H_{-2k}^+(\Gamma)$. The theta lift $\Lambda^M(F,\tau)$ is a harmonic function which transforms like a modular form of weight $1/2 - k$ for ρ_L.*

Proof. By what we have said above, all integrals converge. The transformation behaviour is then obvious. To prove that the lifts are harmonic we use the relations in Lemma 2.3.5, Lemma 2.4.8 and Stokes' theorem as above. $\square$

Remark 3.2.2. Similarly, we can define a theta lift

$$\widetilde{\Lambda}^{\mathrm{M}}(F,\tau) = \begin{cases} R_{1/2,\tau}^{(k+1)/2} \displaystyle\int_{\Gamma\backslash\mathbb{H}} R_{-2k,z}^k F(z)\Theta_{M,0}(\tau,z)d\mu(z), & \text{if } k \text{ is odd,} \\ R_{3/2,\tau}^{k/2} \displaystyle\int_{\Gamma\backslash\mathbb{H}} R_{-2k,z}^k F(z)\Theta_{KM}(\tau,z)d\mu(z), & \text{if } k \text{ is even.} \end{cases}$$

This gives a weakly holomorphic modular form of weight $3/2 + k$ for ρ_L if $k > 0$, see [Alf14, Alf15]. The case $k = 0$, i.e. the Kudla-Millson theta lift, was considered by Bruinier and Funke in [BF06]. It yields a harmonic Maass form of weight $3/2$ for ρ_L which maps to a linear combination of unary theta series of weight $1/2$ under $\xi_{3/2}$.

We now want to show that the regularized Millson theta lift $I^M(F,\tau)$ defined above agrees with $\Lambda^M(F,\tau)$ up to some constant. This connects our work with the work of Alfes-Neumann [Alf15], and it will be useful when we compute the Fourier coefficients of $I^M(F,\tau)$.

Theorem 3.2.3. *Let $k \in \mathbb{Z}_{\geq 0}$ and $F \in H^+_{-2k}(\Gamma)$. Then*

$$I^M(F,\tau) = \left(\left(-\frac{\pi}{N}\right)^{k/2} \prod_{j=0}^{k/2-1} (k-2j)(k+2j+1) \right)^{-1} \Lambda^M(F,\tau),$$

if k is even, and

$$I^M(F,\tau) = \left(-\frac{1}{2\sqrt{N}} \left(-\frac{\pi}{N}\right)^{(k-1)/2} \prod_{j=0}^{(k-1)/2} (k-2j+1)(k+2j) \right)^{-1} \Lambda^M(F,\tau),$$

if k is odd.

Proof. The proof involves several applications of Stokes' theorem. Using the growth estimates of the theta functions given in Proposition 2.4.7 it is straightforward but tedious to verify that all boundary integrals vanish in the limit. We omit these verifications to simplify the exposition.

First let k be even. We consider the expression

$$I_j(F,\tau) = \lim_{T\to\infty} L^{k/2-j}_{1/2-2j,\tau} \int_{M_T} R^{k-2j}_{-2k,z} F(z) \Theta_{M,2j}(\tau,z) y^{-4j} d\mu(z)$$

for $0 \leq j \leq k/2$. By the same arguments as above, it converges to a harmonic function of weight $1/2-k$ for ρ_L which equals $\Lambda^M(F,\tau)$ for $j=0$ and $I^M(F,\tau)$ for $j=k/2$. We split off the innermost lowering operator in τ and the two outermost raising operators in z and apply (a variant of) Lemma 2.3.7 twice to see that $I_j(F,\tau)$ equals

$$\lim_{T\to\infty} L^{k/2-j-1}_{1/2-2j-2,\tau} \int_{M_T} R^{k-2j-2}_{-2k,z} F(z) \overline{L_{-4j-2,z} L_{-4j,z} \overline{L_{1/2-2j,\tau} \Theta_{M,2j}(\tau,z)}} y^{-4j-4} d\mu(z).$$

By Lemma 2.4.11 we have

$$L_{-4j-2,z} L_{-4j,z} \overline{L_{1/2-2j,\tau} \Theta_{M,2j}(\tau,z)} = \frac{\pi}{N} (\Delta_{-4j-4,z} - 8j - 6) \overline{\Theta_{M,2j+2}(\tau,z)}.$$

Using Lemma 2.3.9 we now move the Laplace operator to $R^{k-2k-2}_{-2k,z} F$ in the integral over M_T. Lemma 2.3.5 shows that

$$\Delta_{-4j-4,z} R^{k-2j-2}_{-2k} F = -(k-2j-2)(k+2j+3) R^{k-2j-2}_{-2k,z} F,$$

so together we obtain after a short calculation

$$I_j(F,\tau) = -\frac{\pi}{N}(k-2j)(k+2j+1) I_{j+1}(F,\tau).$$

The formula for even k now follows inductively.

For odd k we first split off the innermost lowering operator in τ and the outermost raising operator in z in $\Lambda^M(F,\tau)$ and apply Lemma 2.3.7 to get

$$\Lambda^M(F,\tau) = -\lim_{T\to\infty} L_{-1/2,\tau}^{(k-1)/2} \int_{M_T} R_{-2k,z}^{k-1} F(z) \overline{L_{0,z}\overline{L_{3/2,\tau}\Theta_{KM}(\tau,z)}} y^{-2} d\mu(z).$$

By Lemma 2.4.11 we have

$$L_{0,z}\overline{L_{3/2,\tau}\Theta_{KM}(\tau,z)} = -\frac{1}{2\sqrt{N}}(\Delta_{-2,z}-2)\overline{\Theta_{M,1}(\tau,z)}.$$

Moving the Laplace operator to $R_{-2k}^{k-1}F$ with Lemma 2.3.9 and using that

$$\Delta_{-2,z} R_{-2k,z}^{k-1} F = -(k-1)(k+2) R_{-2k,z}^{k-1} F$$

we arrive at

$$\Lambda^M(F,\tau) = -\frac{1}{2\sqrt{N}} k(k+1) \lim_{T\to\infty} L_{-1/2,\tau}^{(k-1)/2} \int_{M_T} R_{-2k,z}^{k-1} F(z) \Theta_{M,1}(\tau,z) y^{-2} d\mu(z).$$

Similarly as in the even k case we consider

$$I_j(F,\tau) = \lim_{T\to\infty} L_{-1/2-2j,\tau}^{(k-1)/2-j} \int_{M_T} R_{-2k,z}^{k-1-2j} F(z) \Theta_{M,2j+1}(\tau,z) y^{-4j-2} d\mu(z)$$

for $0 \leq j \leq (k-1)/2$. Note that

$$-\frac{1}{2\sqrt{N}} k(k+1) I_0 = \Lambda^M \quad \text{and} \quad I_{(k-1)/2} = I^M.$$

As above we see that

$$I_j = -\frac{\pi}{N}(k-(2j+1))(k+(2j+1)+1) I_{j+1}.$$

The formula for odd k now follows inductively. □

Remark 3.2.4. In [Alf15] the relation between the Millson lift and the Shintani lift from Proposition 3.1.2 was proven for $k=0$ and under the hypothesis that the constant coefficients $a_\ell^+(0)$ of F vanish at all cusps. It was conjectured there that the relation from Proposition 3.1.2 holds for the lift $\Lambda^M(F,\tau)$ for all $k \geq 0$, but this seems to be difficult to prove directly. Theorem 3.2.3 and Proposition 3.1.2 show that it indeed holds.

3.3 The Fourier expansion of the Millson lift

The Fourier expansions of the Millson lift involves traces of CM values and geodesic cycle integrals of the input function, as well as the so-called complementary traces, which form the principal part and are defined as follows. Let $m \in \mathbb{Q}_{<0}$ and assume that $|m|/N$ is a square, i.e. $m = -Nd^2$ for some $d \in \mathbb{Q}$. Let $F \in H^+_{-2k}(\Gamma)$. For an isotropic line ℓ we let $a^+_\ell(w)$ be the coefficients of the holomorphic part F^+_ℓ of $F_\ell = F|_{-2k}\sigma_\ell$. Let $X \in L_{-Nd^2,h}$. Recall that $\overline{\Gamma}_X$ is trivial and X gives rise to an infinite geodesic $c(X)$. Choosing the orientation of V appropriately, we have

$$\sigma_{\ell_X}^{-1} X = d \begin{pmatrix} 1 & -2r_{\ell_X} \\ 0 & -1 \end{pmatrix}$$

for some $r_{\ell_X} \in \mathbb{Q}$. Note that the geodesic c_X in D is given by

$$c_X = \sigma_{\ell_X}\{z \in \mathbb{H} : \mathrm{Re}(z) = r_{\ell_X}\}.$$

Therefore we call $\mathrm{Re}(c(X)) := r_{\ell_X}$ the real part of $c(X)$. We now define the complementary trace of F by

$$\mathrm{tr}^c_F(-Nd^2, h) = \sum_{X \in \Gamma \backslash L_{-Nd^2,h}} \Bigg(\sum_{w<0} a^+_{\ell_X}(w)(4\pi w)^k e^{2\pi i \,\mathrm{Re}(c(X)) w} + (-1)^{k+1} \sum_{w<0} a^+_{\ell_{-X}}(w)(4\pi w)^k e^{2\pi i \,\mathrm{Re}(c(-X)) w} \Bigg).$$

Let $L_{-Nd^2,h,\ell} = \{X \in L_{-Nd^2,h} : \ell = \ell_X\}$. Repeating the proof of [BF06], Proposition 4.7, we see that the complementary trace can also be written as

$$\begin{aligned} \mathrm{tr}^c_F(-Nd^2, h) = &\sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \nu_\ell(-Nd^2, h) \sum_{w \in \frac{2d}{\beta_\ell}\mathbb{Z}_{<0}} a^+_\ell(w)(4\pi w)^k e^{2\pi i r w} \\ &+ (-1)^{k+1} \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \nu_\ell(-Nd^2, -h) \sum_{w \in \frac{2d}{\beta_\ell}\mathbb{Z}_{<0}} a^+_\ell(w)(4\pi w)^k e^{2\pi i r' w}, \end{aligned} \tag{3.3.1}$$

where $\nu_\ell(-Nd^2, h)$ equals $2d\varepsilon_\ell$ if $L_{-Nd^2,h,\ell} \neq \emptyset$, and 0 otherwise. Further $r = \mathrm{Re}(c(X))$ for any $X \in L_{-Nd^2,h,\ell}$, and $r' = \mathrm{Re}(c(X))$ for any $X \in L_{-Nd^2,-h,\ell}$, if there exist such elements X. In particular, this shows that $\mathrm{tr}^c_F(-Nd^2, h) = 0$ for all but finitely many d.

We are now ready to state the Fourier expansion of the Millson theta lift.

Theorem 3.3.1. *Let $k \in \mathbb{Z}_{\geq 0}$ and let $F \in H^+_{-2k}(\Gamma)$. For $k > 0$ the h-th component of $I^M(F,\tau)$ is given by*

$$\begin{aligned}
&\sum_{m>0} \frac{1}{2\sqrt{m}} \left(\frac{\sqrt{N}}{4\pi\sqrt{m}} \right)^k \left(\operatorname{tr}^+_{R^k_{-2k}F}(m,h) + (-1)^{k+1} \operatorname{tr}^-_{R^k_{-2k}F}(m,h) \right) q^m \\
&+ \sum_{d>0} \frac{1}{2i\sqrt{N}d} \left(\frac{1}{4\pi i d} \right)^k \operatorname{tr}^c_F(-Nd^2,h) q^{-Nd^2} \\
&+ \frac{(-1)^k k!}{2\sqrt{N}\pi^{k+1}} \sum_{\substack{\ell \in \Gamma \backslash \operatorname{Iso}(V) \\ \ell \cap (L+h) \neq \emptyset}} a^+_\ell(0) \frac{\alpha_\ell}{\beta_\ell^{k+1}} \left(\zeta(s+1, k_\ell/\beta_\ell) + (-1)^{k+1} \zeta(s+1, 1-k_\ell/\beta_\ell) \right)\Big|_{s=k} \\
&- \sum_{m<0} \frac{1}{2(4\pi|m|)^{k+1/2}} \overline{\operatorname{tr}_{\xi_{-2k}F}(m,h)} \Gamma\left(1/2+k, 4\pi|m|v\right) q^m,
\end{aligned}$$

where

$$\zeta(s,\rho) = \sum_{\substack{n \geq 0 \\ n+\rho \neq 0}} (n+\rho)^{-s}$$

is the Hurwitz zeta function, and $k_\ell \in \mathbb{Q}$ with $0 \leq k_\ell < \beta_\ell$ is defined by $\sigma_\ell^{-1} h_\ell = \left(\begin{smallmatrix} 0 & k_\ell \\ 0 & 0 \end{smallmatrix}\right)$ for some $h_\ell \in \ell \cap (L+h)$.

For $k = 0$ the h-th component of $I^M(F,\tau)$ is given by the same formula as above but with the additional non-holomorphic terms

$$\sum_{d>0} \frac{1}{4id\sqrt{\pi N}} \sum_{X \in \Gamma \backslash L_{-Nd^2,h}} \left(a^+_{\ell_X}(0) - a^+_{\ell_{-X}}(0) \right) \Gamma\left(1/2, 4\pi N d^2 v\right) q^{-Nd^2}.$$

Note that the first three lines in Theorem 3.3.1 are the holomorphic part of $I^M(F,\tau)$, whereas the fourth line and the additional terms (for $k = 0$) are the non-holomorphic part of $I^M(F,\tau)$. The alternative form of the complementary trace given in (3.3.1) shows that the principal part of $I^M(F,\tau)$ is finite. In particular, this completes the proof of Theorem 3.1.3.

Remark 3.3.2. The Hurwitz zeta function $\zeta(s,\rho)$ is holomorphic for $\operatorname{Re}(s) > 1$ and has a simple pole at $s = 1$ with residue 1 and constant term $-\psi(\rho)$, where $\psi(0) = -\gamma$ (with γ the Euler-Mascheroni constant) and $\psi(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)}$ is the digamma function if $\rho > 0$. Note that $k_\ell = 0$ is equivalent to $h \in L$. Thus for $k = 0$ we get

$$\begin{aligned}
&\left(\zeta(s+1, k_\ell/\beta_\ell) - \zeta(s+1, 1-k_\ell/\beta_\ell) \right)\Big|_{s=0} \\
&\qquad = \psi(1-k_\ell/\beta_\ell) - \psi(k_\ell/\beta_\ell) = \begin{cases} 0, & \text{if } h \in L, \\ \pi\cot(\pi k_\ell/\beta_\ell), & \text{if } h \notin L. \end{cases}
\end{aligned}$$

For $k > 0$ we can simply plug in $s = k$ in the third line of the theorem. For $k_\ell = 0$ we have $\zeta(k+1, 0) = \zeta(k+1, 1) = \zeta(k+1)$, and if $k_\ell \neq 0$ then we could also write

$$\zeta(k+1, k_\ell/\beta_\ell) = \frac{(-1)^{k+1}}{k!}\psi_k(k_\ell/\beta_\ell)$$

with $\psi_k(\rho) = \frac{d^k}{d\rho^k}\psi(\rho)$ the polygamma function.

For the sake of completeness we also state the Fourier expansion of the Shintani lift in our normalization.

Theorem 3.3.3. *Let $k \in \mathbb{Z}_{\geq 0}$ and $G \in S_{2k+2}(\Gamma)$. Then the h-th component of $I^{Sh}(G, \tau)$ is given by*

$$I^{Sh}(G, \tau)_h = -\sqrt{N}\sum_{m>0} \mathrm{tr}_G(-m, h)q^m.$$

Proof. Since $\xi_{-2k,z}$ is surjective, we can choose some $F \in H^+_{-2k}(\Gamma)$ with $\xi_{-2k,z}F = G$. The formula then follows from Theorem 3.3.1 combined with Proposition 3.1.2 and the explicit action of $\xi_{1/2-k,\tau}$ on the Fourier coefficients. Alternatively, the Fourier expansion of $I^{Sh}(G, \tau)$ can be computed using very similar, but much easier calculations as in the proof of Theorem 3.3.1 below. □

Remark 3.3.4. By the same arguments as in [BFI15], Lemma 8.2, we see that

$$-i\sum_{X \in \Gamma\backslash L_{-Nd^2,h}} \left(a^+_{\ell_X}(0) - a^+_{\ell_{-X}}(0)\right) = \frac{1}{N}\sum_{\ell \in \Gamma\backslash \mathrm{Iso}(V)} \varepsilon_\ell a^+_\ell(0) b_{\ell,1}(Nd^2, h),$$

where $b_{\ell,1}(Nd^2, h)$ denotes the (Nd^2, h)-th Fourier coefficient of the theta series $\overline{\Theta_{\ell,1}(\tau)}$. Using this, we can now directly check on the Fourier expansion that $I^M(F, \tau)$ is harmonic, and that it is related to the Shintani lift as stated in Proposition 3.1.2.

3.3.1 Fourier coefficients of positive index

To compute the coefficients of positive index $m > 0$ we use the relation between $I^M(F, \tau)$ and $\Lambda^M(F, \tau)$. The computation of the Fourier coefficients of positive index is completely due to Alfes-Neumann, and can be found in her thesis [Alf15]. We include the calculation for even k for convenience of the reader. The case of odd k is similar. In the case that k is even, the (m, h)-th coefficient of $\Lambda^M(F, \tau)$ is given by

$$C(m, h, v) = L^{k/2}_{1/2,\tau}\int_M R^k_{-2k,z}F(z)\sum_{X \in L_{m,h}} \psi^0_M(X, \tau, z)d\mu(z),$$

where we abbreviated

$$\psi^0_M(X, \tau, z) = vp_X(z)e^{-2\pi vR(X,z)}.$$

By the usual unfolding argument we obtain

$$C(m,h,v) = L_{1/2,\tau}^{k/2} \sum_{X \in \Gamma \backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_X|} \int_{\mathbb{H}} R_{-2k,z}^{k} F(z) \psi_M^0(X,\tau,z) d\mu(z).$$

For $X = \left(\begin{smallmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{smallmatrix}\right) \in L_{m,h}$ and $m > 0$ we have $x_3 \neq 0$ and

$$-2\pi v R(X,z) = 2\pi v (X,X) - \pi v \left(\frac{N(x_3 x - x_1)^2 + q(X)}{\sqrt{N} x_3 y} + \sqrt{N} x_3 y \right)^2,$$

which implies that $\psi_M^0(X,\tau,z)$ is of square-exponential decay in all directions of $\mathbb{H}$. Hence the above integral exists, and the unfolding is justified.

We split $L_{m,h} = L_{m,h}^+ \cup L_{m,h}^-$, and replace X by $-X$ in the sum corresponding to $L_{m,h}^-$, which replaces h by $-h$ and gives a minus sign since $\psi_M^0(X,\tau,z)$ is an odd function of X. Hence it suffices to compute the integral over $\mathbb{H}$ above for fixed $X \in L_{m,h}^+$. Following Katok and Sarnak [KS93] we rewrite it as an integral over $\mathrm{SL}_2(\mathbb{R})$,

$$\int_{\mathbb{H}} R_{-2k,z}^{k} F(z) \psi_M^0(X,\tau,z) d\mu(z) = \int_{\mathrm{SL}_2(\mathbb{R})} R_{-2k,z}^{k} F(gi) \psi_M^0(X,\tau,gi) dg,$$

where we normalize the Haar measure such that the maximal compact subgroup $\mathrm{SO}(2)$ has volume 1. Since $X \in L_{m,h}^+$, there is a $g_1 \in \mathrm{SL}_2(\mathbb{R})$ such that

$$g_1^{-1} X = \sqrt{\frac{m}{N}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\sqrt{2m} X_1(i).$$

Then $g_1 i$ is the Heegner point corresponding to z_X, and replacing g by $g_1 g$ we obtain

$$\int_{\mathrm{SL}_2(\mathbb{R})} R_{-2k,z}^{k} F(gi) \psi_M^0(X,\tau,gi) dg = \int_{\mathrm{SL}_2(\mathbb{R})} R_{-2k,z}^{k} F(g_1 gi) \psi_M^0 \left(-\sqrt{2m} X_1(i), \tau, gi \right) dg.$$

Using the Cartan decomposition of KA^+K of $\mathrm{SL}_2(\mathbb{R})$, where $K = \mathrm{SO}(2)$ and A^+ is set of matrices $a(t) = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ with $t > 0$ (see [Lan75], Chapter 7.2), we find

$$\begin{aligned}
&\int_{\mathrm{SL}_2(\mathbb{R})} R_{-2k,z}^{k} F(g_1 gi) \psi_M^0 \left(-\sqrt{2m} X_1(i), \tau, gi \right) dg \\
&= 4\pi \int_K \int_0^\infty \int_K R_{-2k,z}^{k} F(g_1 k_1 a(t) k_2 i) \psi_M^0 \left(-\sqrt{2m} X_1(i), \tau, k_1 a(t) k_2 i \right) \sinh(2t)\, dk_1\, dt\, dk_2 \\
&= 4\pi \int_1^\infty \left(\int_K R_{-2k,z}^{k} F(g_1 k_1 a(t) i) dk_1 \right) \psi_M^0 \left(-\sqrt{2m} X_1(i), \tau, a(t) i \right) \sinh(2t)\, dt,
\end{aligned}$$

where we used that $k_2 i = i$ and $k_1^{-1} X_1(i) = X_1(i)$, and the integral over k_2 equals 1.

The function

$$H(g) = \int_K R^k_{-2k,z} F(g_1 kgi) dk$$

on G is left and right K-invariant and has the same eigenvalue $\lambda = -k(k+1)$ under the invariant Laplace operator as $R^k_{-2k,z}F(z)$. Thus $H(g)$ is a spherical function of eigenvalue λ (compare [Lan75], Chapter 4.3). By the uniqueness theorem for spherical functions (see [Lan75], Theorem 10 (ii) in Chapter 10.3) it can be written as $H(1)$ times the standard spherical function of eigenvalue λ, which in this case means that

$$H(g) = H(e) P_k(\cosh(2t)),$$

where $e \in G$ is the identity matrix, $g = k_1 a(t) k_2$ and P_k is the usual Legendre polynomial. Note that

$$H(e) = \int_K R^k_{-2k,z} F(g_1 ki) dk = R^k_{-2k,z} F(g_1 i) = R^k_{-2k,z} F(z_X).$$

Summarizing, we obtain

$$\begin{aligned}
&\int_{\mathrm{SL}_2(\mathbb{R})} R^k_{-2k,z} F(g_1 gi) \psi^0_M \left(-\sqrt{2m} X_1(i), \tau, gi\right) dg \\
&= 8\pi \sqrt{m} v R^k_{-2k,z} F(z_X) \int_0^\infty \cosh(2t) \sinh(2t) P_k(\cosh(2t)) e^{-4\pi m v \sinh^2(2t)} dt \\
&= 2\pi \sqrt{m} v R^k_{-2k,z} F(z_X) \int_0^\infty P_k\left(\sqrt{1+t}\right) e^{-4\pi m v x} dx,
\end{aligned}$$

where we replaced $x = \sinh^2(2t)$ in the last step. The latter integral is a Laplace transform, which is computed in equation (7) on page 180 in [EMOT54], so the last line equals

$$\begin{aligned}
&\pi \sqrt{m} v R^k_{-2k,z} F(z_X) (4\pi m v)^{-5/4} W_{1/4,k/2+1/4}(4\pi m v) e^{2\pi m v} \\
&= \frac{1}{2\sqrt{m}} R^k_{-2k,z} F(z_X) \mathcal{W}_{k/2+3/4,1/2}(4\pi m v) e^{2\pi m v},
\end{aligned}$$

where $\mathcal{W}_{s,k}(y) = y^{-k/2} W_{k/2,s-1/2}(y)$ is the $\mathcal{W}$-Whittaker function. Using (13.1.33) and

(13.4.23) in [AS84] it is easy to show that

$$\begin{aligned}
&L_{1/2}^{k/2}\left(\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}}(4\pi mv)e(mu)\right)\\
&=\left(\frac{1}{4\pi m}\right)^{k/2}\prod_{j=0}^{k/2-1}\left(\frac{k+1}{2}+j\right)\left(j-\frac{k}{2}\right)\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}-k}(4\pi mv)e(mu)\\
&=\left(\frac{1}{4\pi m}\right)^{k/2}\prod_{j=0}^{k/2-1}\left(\frac{k+1}{2}+j\right)\left(j-\frac{k}{2}\right)e^{2\pi i m\tau}.
\end{aligned}$$

Taking everything together, we finally obtain that $C(m,h,v)$ equals

$$\frac{1}{2\sqrt{m}}\left(\frac{1}{4\pi m}\right)^{k/2}\prod_{j=0}^{k/2-1}\left(\frac{k+1}{2}+j\right)\left(j-\frac{k}{2}\right)\left(\mathrm{tr}^{+}_{R^k_{-2k}F}(m,h)-\mathrm{tr}^{+}_{R^k_{-2k}F}(m,-h)\right).$$

Note that

$$\mathrm{tr}^{+}_{R^k_{-2k}F}(m,-h)=\mathrm{tr}^{-}_{R^k_{-2k}F}(m,h).$$

Combining this with Theorem 3.2.3 we obtain the formula for the coefficients of positive index.

3.3.2 Fourier coefficients of negative index

The computation of the Fourier coefficients of negative index was conducted in joint work with Alfes-Neumann.

For $m<0$ the (m,h)-th coefficient of $I^M(F,\tau)$ is given by

$$C(m,h,v)=\sum_{X\in\Gamma\backslash L_{m,h}}\lim_{T\to\infty}\int_{M_T}F(z)\sum_{\gamma\in\Gamma_X\backslash\Gamma}\psi^0_{M,k}(\gamma X,\tau,z)y^{-2k}d\mu(z),$$

where

$$\psi^0_{M,k}(X,\tau,z)=v^{k+1}p_X(z)Q_X^k(\bar{z})e^{-2\pi vR(X,z)}.$$

We compute the individual summands for fixed $X\in L_{m,h}$.

The computation follows similar arguments as in the proof of Theorem 4.5 in [BF06]. First, a short calculation using the rules (2.4.2) shows that the function

$$\eta(X,\tau,z)=C_k v^{k+1}Q_X^{-k-1}(z)\frac{\partial^k}{\partial v^k}\left(v^{-1}e^{-2\pi vR(X,z)}\right),\qquad C_k=\frac{\sqrt{N}(2N)^k}{(-2\pi)^{k+1}},\tag{3.3.2}$$

satisfies

$$\xi_{2k+2,z}\eta(X,\tau,z)=\overline{\psi^0_{M,k}(X,\tau,z)}.$$

Writing $e^{-2\pi v R(X,z)} = v\int_{2\pi R(X,z)}^{\infty} e^{-vt}dt$ and differentiating k times under the integral, we can also rewrite η as

$$\eta(X,\tau,z) = (-1)^k C_k Q_X^{-k-1}(z)\Gamma(k+1, 2\pi v R(X,z)). \tag{3.3.3}$$

Using Stokes' theorem in the form given in Lemma 2.3.7, we obtain

$$\lim_{T\to\infty}\int_{M_T} F(z)\sum_{\gamma\in\Gamma_X\backslash\Gamma}\psi^0_{M,k}(\gamma X,\tau,z)y^{-2k}d\mu(z) \tag{3.3.4}$$

$$= -\lim_{T\to\infty}\int_{M_T}\overline{\xi_{-2k,z}F(z)}\sum_{\gamma\in\Gamma_X\backslash\Gamma}\eta(\gamma X,\tau,z)y^{2k+2}d\mu(z) \tag{3.3.5}$$

$$-\lim_{T\to\infty}\int_{\partial M_T} F(z)\sum_{\gamma\in\Gamma_X\backslash\Gamma}\eta(\gamma X,\tau,z)dz. \tag{3.3.6}$$

Since $\xi_{-2k,z}F$ is a cusp form, we can write the limit of the first integral on the right-hand side as an integral over M.

The integral over M

We first compute the complex conjugate of the integral over M on the right-hand side. Since $Q(X) = m < 0$ we can find some matrix $g \in \mathrm{SL}_2(\mathbb{R})$ such that

$$X' := g^{-1}X = \sqrt{\frac{|m|}{N}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Replacing z by gz and using the unfolding argument, we find

$$\int_M \xi_{-2k,z}F(z)\overline{\sum_{\gamma\in\Gamma_X\backslash\Gamma}\eta(\gamma X,\tau,z)}y^{2k+2}d\mu(z) = \int_{\Gamma_{X'}\backslash\mathbb{H}}\xi_{-2k,z}F_g(z)\overline{\eta(X',\tau,z)}y^{2k+2}d\mu(z),$$

where $F_g = F|_{-2k}g$.

If $|m|/N$ is not a square then $\overline{\Gamma}_X$ is infinite cyclic and

$$\Gamma_{X'} = g^{-1}\Gamma_X g = \left\{\pm\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}^n : n\in\mathbb{Z}\right\}$$

for some $\varepsilon > 1$. On the other hand, if $|m|/N$ is a square then $\overline{\Gamma}_{X'}$ is trivial, so $\Gamma_{X'}\backslash\mathbb{H} = \mathbb{H}$. Here we only explain the non-square case since the other case is very similar. As a fundamental domain for $\Gamma_{X'}\backslash\mathbb{H}$ we can take the horizontal strip $\{z\in\mathbb{H} : 1\le y<\varepsilon^2\}$.

Using the explicit formula

$$\overline{\eta(X',\tau,z)} = (-1)^k C_k \left(-2\sqrt{|m|N}\bar{z}\right)^{-k-1} \Gamma\left(k+1, 4\pi|m|v\left(\frac{x^2}{y^2}+1\right)\right), \tag{3.3.7}$$

and replacing $\frac{x}{y}$ by t in the integral over x, we find that the complex conjugate of (3.3.5) equals

$$\frac{(-1)^{k+1}C_k v^{k+1}}{(-2\sqrt{|m|N})^{k+1}} \int_{-\infty}^{\infty}\int_1^{\varepsilon^2} (t+i)^{k+1} y^k \xi_{-2k,z}F_g(y(t+i))dy \, \frac{\Gamma\left(k+1, 4\pi|m|v\left(t^2+1\right)\right)}{(t^2+1)^{k+1}} dt.$$

The inner integral is the contour integral of the holomorphic function $z^k\xi_{-2k,z}F_g(z)$ along the line $y(t+i), y \in (1,\varepsilon^2)$. Using $\xi_{-2k,z}F_g(\varepsilon^2 z) = \varepsilon^{-2k-2}\xi_{-2k,z}F_g(z)$ it is easily seen by Cauchy's theorem that the inner integral does in fact not depend on t. Thus the double integral simplifies to

$$\frac{(-1)^{k+1}C_k v^{k+1} i^{k+1}}{(-2\sqrt{|m|N})^{k+1}} \int_1^{\varepsilon^2} \xi_{-2k,z}F_g(iy)y^k dy \cdot \int_{-\infty}^{\infty} \frac{\Gamma\left(k+1, 4\pi|m|v\left(t^2+1\right)\right)}{(t^2+1)^{k+1}} dt.$$

Plugging in the definition of the incomplete Gamma function, interchanging the order of integration and using $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$, we compute

$$\int_{-\infty}^{\infty} \frac{\Gamma\left(k+1, 4\pi|m|v\left(t^2+1\right)\right)}{(t^2+1)^{k+1}} dt = \sqrt{\pi}\Gamma(1/2+k, 4\pi|m|v).$$

If we put everything together and recall the definition of the cycle integral $\mathcal{C}(\xi_{-2k,z}F,X)$, we see that (3.3.5) equals

$$-\frac{1}{2(4\pi|m|)^{k+1/2}}\overline{\mathcal{C}(\xi_{-2k,z}F,X)}\Gamma(1/2+k,4\pi|m|v).$$

The boundary integral

We now consider the limit of the boundary integral in (3.3.6). By the definition of the truncated curve M_T we find

$$-\int_{\partial M_T} F(z) \sum_{\gamma\in\Gamma_X\backslash\Gamma} \eta(\gamma X,\tau,z)dz = \sum_{\ell\in\Gamma\backslash\operatorname{Iso}(V)} \int_{z=iT}^{\alpha_\ell+iT} F_\ell(z) \sum_{\gamma\in\Gamma_X\backslash\Gamma} \eta(\sigma_\ell^{-1}\gamma X,\tau,z)dz,$$

where $F_\ell = F|_{-2k}\sigma_\ell$. As in the proof of Lemma 5.2 in [BF06] we see that for each isotropic line ℓ the integral vanishes in the limit unless X is orthogonal to ℓ and $\gamma\in\Gamma_\ell$, which can only happen if $|m|/N$ is a square and $\ell=\ell_X$ or $\ell=\ell_{-X}$. In particular, if $|m|/N$ is not a square, then the whole boundary integral vanishes. On the other hand, if $|m|/N$ is a square, we obtain

$$-\int_{\partial M_T} F(z) \sum_{\gamma\in\Gamma_X\backslash\Gamma} \eta(\gamma X,\tau,z)dz = \int_{z=iT}^{\alpha_{\ell_X}+iT} F_{\ell_X}(z) \sum_{\gamma\in\overline{\Gamma}_{\ell_X}} \eta(\sigma_{\ell_X}^{-1}\gamma X,\tau,z)dz, \qquad (3.3.8)$$
$$+\int_{z=iT}^{\alpha_{\ell_{-X}}+iT} F_{\ell_{-X}}(z) \sum_{\gamma\in\overline{\Gamma}_{\ell_{-X}}} \eta(\sigma_{\ell_{-X}}^{-1}\gamma X,\tau,z)dz.$$

We only compute the first integral on the right-hand side since the second one can be computed in the same way if we first replace X by $-X$, which gives a factor $(-1)^{k+1}$.

Let $\ell=\ell_X$ for brevity. Choosing the orientation of V appropriately, we can assume that

$$X' := \sigma_\ell^{-1}.X = \sqrt{\frac{|m|}{N}}\begin{pmatrix}1 & -2r_\ell\\ 0 & -1\end{pmatrix}$$

for some $r_\ell\in\mathbb{Q}$. Then the first summand in (3.3.8) equals

$$\int_{z=iT}^{\alpha_\ell+iT} F_\ell(z) \sum_{\gamma\in\sigma_\ell^{-1}\overline{\Gamma}_\ell\sigma_\ell} \eta(\gamma X',\tau,z)dz.$$

Recall that $\sigma_\ell^{-1}\overline{\Gamma}_\ell\sigma_\ell$ consists of the matrices $\left(\begin{smallmatrix}1 & \alpha_\ell n\\ 0 & 1\end{smallmatrix}\right)$ with $n\in\mathbb{Z}$. Using the definition of η, we see that the first summand in (3.3.8) equals

$$\int_{z=iT}^{\alpha_\ell+iT} F_\ell(z)\sum_{n\in\mathbb{Z}}\eta\left(\sqrt{\frac{|m|}{N}}\begin{pmatrix}1 & 2(\alpha_\ell n-r_\ell)\\ 0 & -1\end{pmatrix},\tau,z\right)dz$$
$$=\frac{C_k v^{k+1}}{(-2\sqrt{|m|N})^{k+1}}\frac{\partial^k}{\partial v^k}v^{-1}e^{-4\pi|m|v}$$
$$\times\int_{z=iT}^{\alpha_\ell+iT} F_\ell(z)\sum_{n\in\mathbb{Z}}(z+\alpha_\ell n-r_\ell)^{-k-1}e^{-4\pi|m|v\frac{(x+\alpha_\ell n-r_\ell)^2}{y^2}}dz.$$

For a function $g(t)$ on $\mathbb{R}$ we let $\hat{g}(w)=\int_{-\infty}^{\infty}g(t)e^{2\pi itw}dt$ be its Fourier transform. Using Poisson summation we can rewrite the inner sum as

$$\sum_{n\in\mathbb{Z}}(z+\alpha_\ell n-r_\ell)^{-k-1}e^{-4\pi|m|v\frac{(x+\alpha_\ell n-r_\ell)^2}{y^2}}$$
$$=\frac{1}{\alpha_\ell}\sum_{w\in\frac{1}{\alpha_\ell}\mathbb{Z}}e^{-2\pi iw(x-r_\ell)}\int_{-\infty}^{\infty}(t+iy)^{-k-1}e^{-4\pi|m|v\frac{t^2}{y^2}}e^{2\pi iwt}dt,$$

where we replaced $t=(x+\alpha_\ell n-r_\ell)$. The required Fourier transform is computed in the next lemma. We let $a = 2\sqrt{\pi|m|v}/y$ and $b = y$ for brevity.

Lemma 3.3.5. *For $a, b \neq 0$ and $k \in \mathbb{Z}, k \geq 0$, the Fourier transform of*

$$h_k(t) = (t + ib)^{-k-1} e^{-a^2t^2}$$

is given by

$$\begin{aligned}\widehat{h}_k(w) = &-\frac{i^{k+1}}{k!}\pi e^{a^2b^2} e^{2\pi bw}\Bigg(\operatorname{erfc}(ab + \pi w/a) \sum_{j=0}^{k} \binom{k}{j} (2\pi w)^{k-j}(-ia)^j H_j(iab) \\ &+ e^{-(ab+\pi w/a)^2} \frac{2}{\sqrt{\pi}} \sum_{j=1}^{k} \binom{k}{j} (2\pi w)^{k-j}(-a)^j \sum_{\ell=0}^{j-1} \binom{j}{\ell} i^\ell H_\ell(iab) H_{j-\ell-1}(ab + \pi w/a)\Bigg),\end{aligned}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

is the standard complementary error function and

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

is the n-th Hermite polynomial.

Proof. Since

$$(t + ib)^{-k-1} e^{-a^2t^2} = \frac{i^k}{k!} \left(\frac{\partial^k}{\partial b^k} (t + ib)^{-1} \right) e^{-a^2t^2},$$

the formula for $\widehat{h}_k$ follows from the one for $\widehat{h}_0$ and Leibniz's rule. Thus it suffices to prove that the Fourier transform of

$$h_0(t) = (t + ib)^{-1} e^{-a^2t^2} = (t - ib)\frac{e^{-a^2t^2}}{t^2 + b^2}$$

is given by

$$\widehat{h}_0(w) = -i\pi e^{a^2b^2} e^{2\pi bw} \operatorname{erfc}(ab + \pi w/a).$$

Using the well known facts that the Fourier transforms of $e^{-a^2t^2}$ and $\frac{1}{t^2+b^2}$ are given by $\frac{\sqrt{\pi}}{a} e^{-\pi^2w^2/a^2}$ and $\frac{\pi}{b} e^{-2\pi b|w|}$, respectively, and that the Fourier transform of a product of two functions is the convolution of the individual transforms, we see that the Fourier transform of $f(t) = \frac{e^{-a^2t^2}}{t^2+b^2}$ is given by

$$\begin{aligned}\widehat{f}(w) &= \frac{\pi^{3/2}}{ab}\int_{-\infty}^{\infty} e^{-\pi^2x^2/a^2}e^{-2\pi b|w-x|}dx\\ &= \frac{\pi^{3/2}}{ab}e^{2\pi bw}\int_{w}^{\infty} e^{-\pi^2x^2/a^2}e^{-2\pi bx}dx + \frac{\pi^{3/2}}{ab}e^{-2\pi bw}\int_{-w}^{\infty} e^{-\pi^2x^2/a^2}e^{-2\pi bx}dx\\ &= \frac{\pi}{2b}e^{a^2b^2}\left(e^{2\pi bw}\operatorname{erfc}(ab+\pi w/a) + e^{-2\pi bw}\operatorname{erfc}(ab-\pi w/a)\right).\end{aligned}$$

Since the Fourier transform of $tf(t)$ is given by $-\frac{i}{2\pi}\frac{d}{dw}\widehat{f}(w)$, we obtain

$$\widehat{(tf)}(w) = -\frac{i\pi}{2}e^{a^2b^2}\left(e^{2\pi bw}\operatorname{erfc}(ab+\pi w/a) - e^{-2\pi bw}\operatorname{erfc}(ab-\pi w/a)\right).$$

Using $\widehat{h}_0 = \widehat{(tf)} - ib\widehat{f}$ we get the stated formula. □

Let $a_\ell^+(w)$ and $a_\ell^-(w)$ denote the Fourier coefficients of F_ℓ. Using the above lemma with $a = 2\sqrt{\pi|m|v}/y$ and $b = y$, we find that the right-hand side of (3.3.8) is equal to

$$\begin{aligned}&\frac{-C_k v^{k+1} i^{k+1}\pi}{(-2\sqrt{|m|N})^{k+1}k!}\frac{\partial^k}{\partial v^k}v^{-1}\lim_{T\to\infty}\sum_{w\in\frac{1}{\alpha_\ell}\mathbb{Z}}(a_\ell^+(w) + a_\ell^-(w)\Gamma(1+2k,4\pi|w|T))e^{2\pi i r_\ell w}\\ &\times\Bigg(\operatorname{erfc}\left(2\sqrt{\pi|m|v} + T\sqrt{\pi}w/(2\sqrt{|m|v})\right)\\ &\qquad\sum_{j=0}^{k}\binom{k}{j}(2\pi w)^{k-j}\left(-2i\sqrt{\pi|m|v}/T\right)^j H_j\left(2i\sqrt{\pi|m|v}\right)\\ &\quad + e^{-\left(2\sqrt{\pi|m|v}+T\sqrt{\pi}w/(2\sqrt{|m|v})\right)^2}\frac{2}{\sqrt{\pi}}\sum_{j=1}^{k}\binom{k}{j}(2\pi w)^{k-j}\left(-2\sqrt{\pi|m|v}/T\right)^j\\ &\qquad\sum_{\ell=0}^{j-1}\binom{j}{\ell}i^\ell H_\ell\left(2i\sqrt{\pi|m|v}\right)H_{j-\ell-1}\left(2\sqrt{\pi|m|v} + T\sqrt{\pi}w/(2\sqrt{|m|v})\right)\Bigg).\end{aligned}$$

Note that $\operatorname{erfc}(x) = O(e^{-x^2})$ as $x \to +\infty$ and $\lim_{x\to-\infty}\operatorname{erfc}(x) = 2$. Further, the incomplete Gamma function is of linear exponential decay.

For $k = 0$ the last three lines disappear (i.e., they have to replaced by 1), and the summands for $w > 0$ vanish as $T \to \infty$. Thus all that remains in the limit is

$$-\frac{i}{2\sqrt{|m|}}\sum_{w<0}a_\ell^+(w)e^{2\pi i r_\ell w} - \frac{i}{4\sqrt{|m|}}a_\ell^+(0)\operatorname{erfc}(2\sqrt{\pi|m|v}),$$

in this case. Note that $\sqrt{\pi}\operatorname{erfc}(2\sqrt{\pi|m|v}) = \Gamma(\frac{1}{2}, 4\pi|m|v)$.

For $k > 0$ all summands for $w \geq 0$ vanish in the limit. Further, the summands for $1 \leq j \leq k$ in the third row, and the two last rows vanish as $T \to \infty$. Thus we are left with

$$\frac{(-i)^{k+1}}{2\sqrt{|m|}} \left(\frac{\sqrt{N}}{4\pi\sqrt{|m|}} \right)^k \sum_{w<0} a_\ell^+(w)(4\pi w)^k e^{2\pi i r_\ell w},$$

if $k > 0$. Here we used $v^{k+1} \frac{\partial^k}{\partial v^k} v^{-1} = (-1)^k k!$.

3.3.3 Fourier coefficients of index 0

These coefficients were also computed jointly with Alfes-Neumann. We now want to compute

$$C(0, h, v) = \lim_{T\to\infty} \int_{M_T} F(z) \sum_{X \in L_{0,h}} \psi^0_{M,k}(X, \tau, z) y^{-2k} d\mu(z),$$

where

$$L_{0,h} = \{X \in L + h : Q(X) = 0\}.$$

Note that the sum over X is now infinite. Further, we have $\psi^0_{M,k}(0, \tau, z) = 0$ so we can leave out the summand for $X = 0$. The computation for $Q(X) = 0$ is quite similar to the one for $Q(X) < 0$ above, so we skip some arguments. Using the function $\eta(X, \tau, z)$ defined in (3.3.2) and Stokes' theorem we get

$$\begin{aligned} C(0, h, v) = &- \lim_{T\to\infty} \int_{M_T} \overline{\xi_{-2k,z} F(z)} \sum_{\substack{X \in L_{0,h} \\ X \neq 0}} \eta(X, \tau, z) y^{2k+2} d\mu(z) \\ &- \lim_{T\to\infty} \int_{\partial M_T} F(z) \sum_{\substack{X \in L_{0,h} \\ X \neq 0}} \eta(X, \tau, z) dz. \end{aligned}$$

Since $\xi_{-2k,z}F$ is a cusp form, we can write the first integral on the right-hand side as an integral over M.

For each isotropic line $\ell \in \mathrm{Iso}(V)$ we choose a positively oriented primitive vector $X_\ell \in \ell \cap L$. If $\ell \cap (L + h) \neq \emptyset$ we can fix some vector $h_\ell \in \ell \cap (L + h)$ and write $\ell \cap (L + h) = \mathbb{Z}X_\ell + h_\ell$. Note that $\sigma_\ell^{-1}(nX_\ell + h_\ell) = \left(\begin{smallmatrix} 0 & n\beta_\ell + k_\ell \\ 0 & 0 \end{smallmatrix}\right)$ for some $k_\ell \in \mathbb{Q}$.

We now parametrize the set $L_{0,h} \setminus \{0\}$ by the points $nX_\ell + h_\ell$, where ℓ runs through all isotropic lines with $\ell \cap (L + h) \neq \emptyset$ and n runs through $\mathbb{Z}$ such that $n\beta_\ell + k_\ell \neq 0$.

The integral over M

Using the above parametrization for $L_{0,h}\setminus\{0\}$ the integral over M in $C(0,h,v)$ becomes

$$-\sum_{\substack{\ell\in\Gamma\setminus\operatorname{Iso}(V)\\ \ell\cap(L+h)\neq\emptyset}}\int_M \overline{\xi_{-2k,z}F(z)}\sum_{\substack{n\in\mathbb{Z}\\ n\beta_\ell+k_\ell\neq 0}}\sum_{\gamma\in\Gamma_\ell\setminus\Gamma}\eta(\gamma(nX_\ell+h_\ell),\tau,z)y^{2k+2}d\mu(z).$$

Replacing z by $\sigma_\ell z$ and using the unfolding argument, we get

$$-\sum_{\substack{\ell\in\Gamma\setminus\operatorname{Iso}(V)\\ \ell\cap(L+h)\neq\emptyset}}\int_0^\infty\int_0^{\alpha_\ell}\overline{\xi_{-2k,z}F_\ell(z)}\sum_{\substack{n\in\mathbb{Z}\\ n\beta_\ell+k_\ell\neq 0}}\eta\left(\begin{pmatrix}0 & n\beta_\ell+k_\ell\\ 0 & 0\end{pmatrix},\tau,z\right)y^{2k+2}\frac{dx\ dy}{y^2}$$

where $F_\ell = F|_{-2k}\sigma_\ell$. Explicitly, we have

$$\eta\left(\begin{pmatrix}0 & n\beta_\ell+k_\ell\\ 0 & 0\end{pmatrix},\tau,z\right) = C_k v^{k+1}(-N(n\beta_\ell+k_\ell))^{-k-1}\frac{\partial^k}{\partial v^k}\left(v^{-1}e^{-\pi vN\frac{(n\beta_\ell+k_\ell)^2}{y^2}}\right), \tag{3.3.9}$$

which is independent of x. Therefore the integral over x picks out the constant coefficient of $\overline{\xi_{-2k,z}F_\ell}$, which is 0 since $\xi_{-2k,z}F_\ell$ is a cusp form. Thus the integral over M vanishes.

The boundary integral

Plugging in the definition of the truncated curve (2.2.1), the boundary integral is given by

$$\lim_{T\to\infty}\sum_{\substack{\ell\in\Gamma\setminus\operatorname{Iso}(V)\\ \ell\cap(L+h)\neq\emptyset}}\sum_{\ell'\in\Gamma\setminus\operatorname{Iso}(V)}\int_{z=iT}^{\alpha_{\ell'}+iT}F_{\ell'}(z)\sum_{\substack{n\in\mathbb{Z}\\ n\beta_\ell+k_\ell\neq 0}}\sum_{\gamma\in\Gamma_\ell\setminus\Gamma}\eta\left(\sigma_{\ell'}^{-1}\gamma\sigma_\ell\begin{pmatrix}0 & n\beta_\ell+k_\ell\\ 0 & 0\end{pmatrix},\tau,z\right)dz.$$

It can be seen as in the proof of Lemma 5.2 in [BF06] that in the limit only the contributions for $\ell'=\ell$ and $\gamma\in\Gamma_\ell$ remain, so we get

$$\lim_{T\to\infty}\sum_{\substack{\ell\in\Gamma\setminus\operatorname{Iso}(V)\\ \ell\cap(L+h)\neq\emptyset}}\int_{z=iT}^{\alpha_\ell+iT}F_\ell(z)\sum_{\substack{n\in\mathbb{Z}\\ n\beta_\ell+k_\ell\neq 0}}\eta\left(\begin{pmatrix}0 & n\beta_\ell+k_\ell\\ 0 & 0\end{pmatrix},\tau,z\right)dz.$$

Using the explicit form (3.3.9) of η and carrying out the integral this becomes

$$\frac{C_k v^{k+1}}{(-N)^{k+1}}\frac{\partial^k}{\partial v^k}v^{-1}\sum_{\substack{\ell\in\Gamma\setminus\operatorname{Iso}(V)\\ \ell\cap(L+h)\neq\emptyset}}\alpha_\ell a_\ell^+(0)\lim_{T\to\infty}\sum_{\substack{n\in\mathbb{Z}\\ n\beta_\ell+k_\ell\neq 0}}(n\beta_\ell+k_\ell)^{-k-1}e^{-\pi vN\frac{(n\beta_\ell+k_\ell)^2}{T^2}}.$$

If $k_\ell/\beta_\ell \in \mathbb{Z}$, we can shift the summation index by k_ℓ/β_ℓ and see that the terms with n and $-n$ cancel if k is even and add up if k is odd, so in this case the limit of the sum over n is 0 if k is even or $2\beta_\ell^{-k-1}\zeta(k+1)$ if k is odd. On the other hand, $k_\ell/\beta_\ell \in \mathbb{Z}$ is only possible if h_ℓ is an integral multiple of X_ℓ and hence in L, i.e. this only happens for $h = 0 \bmod L$.

Now let $h \neq 0 \bmod L$ and thus $k_\ell/\beta_\ell \notin \mathbb{Z}$. For $k > 0$ we can interchange the sum and the limit by the dominated convergence theorem. Splitting the sum into $n \geq 0$ and $n < 0$ and replacing n by $1-n$ in the second part, we obtain

$$\begin{aligned}\lim_{T\to\infty} \sum_{n\in\mathbb{Z}} &(n\beta_\ell + k_\ell)^{-k-1} e^{-\pi v N \frac{(n\beta_\ell+k_\ell)^2}{T^2}} \\ &= \beta_\ell^{-k-1}\left(\zeta(k+1, k_\ell/\beta_\ell) + (-1)^{k+1}\zeta(k+1, 1-k_\ell/\beta_\ell)\right),\end{aligned}$$

where $\zeta(s,\rho) = \sum_{n\geq 0}(n+\rho)^{-s}$ denotes the Hurwitz zeta function. For $k = 0$ we first reorder the sum as

$$\begin{aligned}\sum_{n\in\mathbb{Z}} (n\beta_\ell + k_\ell)^{-k-1} e^{-\pi N v \frac{(n\beta_\ell+k_\ell)^2}{T^2}} &= k_\ell^{-1} e^{-\pi v N \frac{k_\ell^2}{T^2}} \\ + \beta_\ell^{-1} \sum_{n>0} &\left((n + k_\ell/\beta_\ell)^{-1} e^{-\pi v N \frac{(n\beta_\ell+k_\ell)^2}{T^2}} + (-n + k_\ell/\beta_\ell)^{-1} e^{-\pi v N \frac{(-n\beta_\ell+k_\ell)^2}{T^2}} \right).\end{aligned}$$

Now using dominated convergence again, this goes to

$$\beta_\ell^{-1}\left(\sum_{n>0}\left((n+k_\ell/\beta_\ell)^{-1} + (-n+k_\ell/\beta_\ell)^{-1}\right) + (k_\ell/\beta_\ell)^{-1}\right) = \beta_\ell^{-1}\pi\cot(\pi k_\ell/\beta_\ell)$$

as $T \to \infty$. Note that $\frac{C_k v^{k+1}}{(-N)^{k+1}} \frac{\partial^k}{\partial v^k} v^{-1} = \frac{(-1)^k k!}{2\sqrt{N}\pi^{k+1}}$. This completes the calculation of $C(0,h,v)$, and of the Fourier expansion of the Millson lift.

3.4 The twisted Millson theta lift

Using the twisted theta functions from Section 2.4.4 we construct twisted analogs of the lifts considered above. Throughout this section, we let $\Gamma = \Gamma_0(N)$ and we let L be the special lattice from Section 2.2.5 corresponding to $\Gamma_0(N)$. Further, we let Δ be a fundamental discriminant. Recall from Section 2.4.4 that $L^{(\Delta)}$ denotes be the lattice ΔL with the quadratic form $Q_\Delta(X) = \frac{1}{|\Delta|}Q(X)$. Its discriminant group is $L'/L^{(\Delta)}$. We let $\Gamma^{(\Delta)}$ be the subgroup of Γ which acts on $L^{(\Delta)}$ and fixes the classes of $L'/L^{(\Delta)}$. Further, recall that $\tilde{\rho}_L$ denotes ρ_L if $\Delta > 0$ and ρ_L^* if $\Delta < 0$. Throughout, a superscript (Δ) indicates that the corresponding quantity or object is taken with respect to $L^{(\Delta)}$ and $\Gamma^{(\Delta)}$.

For a harmonic weak Maass form $F \in H^+_{-2k}(\Gamma_0(N))$ we define the twisted Millson theta lift by

$$I^M_{\Delta,r}(F,\tau) = \lim_{T\to\infty} \int_{M_T} F(z)\Theta_{M,k,\Delta,r}(\tau,z) y^{-2k} d\mu(z),$$

where

$$\Theta_{M,k,\Delta,r}(\tau,z) = \sum_{h\in L'/L} \left\langle \psi_{\Delta,r}(\mathfrak{e}_h), \overline{\Theta^{(\Delta)}_{M,k}(\tau,z)} \right\rangle \mathfrak{e}_h$$

denotes the (Δ,r)-th twisted Millson theta function, compare Section 2.4.4. Analogously we obtain the twisted Shintani lift $I^{Sh}_{\Delta,r}(F,\tau)$. The twisted theta lifts have the same mapping properties as their untwisted versions (see Theorem 3.1.3), with ρ_L replaced by $\tilde{\rho}_L$. Further, we obtain the following generalization of Proposition 3.1.2.

Proposition 3.4.1. *For $F \in H^+_0(\Gamma)$ we have*

$$\xi_{1/2,\tau}(I^M_{\Delta,r}(F,\tau)) = -\frac{\sqrt{|\Delta|}}{2\sqrt{N}} I^{Sh}_{\Delta,r}(\xi_{0,z}F,\tau) + \frac{1}{2N} \sum_{\ell\in\Gamma\backslash \operatorname{Iso}(V)} \varepsilon_\ell \overline{a^+_\ell(0)\Theta_{\ell,1,\Delta,r}(\tau)},$$

and for $k \in \mathbb{Z}_{>0}$ and $F \in H^+_{-2k}(\Gamma)$ we have

$$\xi_{1/2-k,\tau}(I^M_{\Delta,r}(F,\tau)) = -\frac{\sqrt{|\Delta|}}{2\sqrt{N}} I^{Sh}_{\Delta,r}(\xi_{-2k,z}F,\tau).$$

Proof. Let us assume $k = 0$ for simplicity. Following the approach of [AE13] we write

$$I^M_{\Delta,r}(F,\tau) = \frac{1}{[\Gamma:\Gamma^{(\Delta)}]} \sum_{h\in L'/L} \left\langle \psi_{\Delta,r}(\mathfrak{e}_h), \overline{I^M\left(F,\tau,L^{(\Delta)},\Gamma^{(\Delta)}\right)} \right\rangle \mathfrak{e}_h, \tag{3.4.1}$$

where

$$I^M\left(F,\tau,L^{(\Delta)},\Gamma^{(\Delta)}\right) = \lim_{T\to\infty} \int_{M_T^{(\Delta)}} F(z)\Theta^{(\Delta)}_{M,k}(\tau,z) d\mu(z)$$

is the untwisted Millson theta lift for the lattice $L^{(\Delta)}$ and $M_T^{(\Delta)}$ is the truncated version of the curve $M^{(\Delta)} = \Gamma^{(\Delta)}\backslash\mathbb{H}$. By Proposition 3.1.2 we have

$$\begin{aligned}
&\xi_{1/2,\tau}\left(I^M\left(F,\tau,L^{(\Delta)},\Gamma^{(\Delta)}\right)\right)\\
&= -\frac{\sqrt{|\Delta|}}{2\sqrt{N}} I^{Sh}\left(\xi_{0,z}F,\tau,L^{(\Delta)},\Gamma^{(\Delta)}\right) + \frac{|\Delta|}{2N} \sum_{\ell\in\Gamma^{(\Delta)}\backslash \operatorname{Iso}(V)} \varepsilon^{(\Delta)}_\ell \overline{a^+_\ell(0)\Theta^{(\Delta)}_{\ell,1}(\tau)}.
\end{aligned}$$

Now a short calculation, using

$$\varepsilon^{(\Delta)}_\ell = \frac{[\Gamma_\ell : (\Gamma^{(\Delta)})_\ell]}{|\Delta|}\varepsilon_\ell,$$

the decomposition (3.4.1) and the analogous decomposition for $I^{Sh}_{\Delta,r}(\xi_{0,z}F,\tau)$, yields the result. $\square$

This relation gives an interesting criterion for the non-vanishing of the twisted L-function of a newform at the critical point. The corresponding result for square free N and odd k was proven in the thesis [Alf15].

Theorem 3.4.2. *Let $F \in H^+_{-2k}(\Gamma)$, with vanishing constant terms at all cusps if $k=0$, such that $G = \xi_{-2k}F \in S_{2k+2}(\Gamma)$ is a normalized newform. For $\Delta < 0$ with $(\Delta, N) = 1$ the lift $I^M_{\Delta,r}(F,\tau)$ is weakly holomorphic if and only if $L(G,\chi_\Delta,k+1) = 0$.*

Proof. By the last proposition, $I^M_{\Delta,r}(F,\tau)$ is weakly holomorphic if and only if the Shintani lift $I^{Sh}_{\Delta,r}(G,\tau)$ vanishes. Since G is a normalized newform, Corollary 2 in Section II.4. of [GKZ87] shows that the square of the absolute value of the D-th coefficient ($D < 0$ with $(D,N) = 1$ a fundamental discriminant) of $I^{Sh}_{\Delta,r}(G,\tau)$ (viewed as a Jacobi form) is up to non-zero factors given by $L(G,\chi_\Delta,k+1)L(G,\chi_D,k+1)$. If $L(G,\chi_\Delta,k+1) = 0$, then all fundamental coefficients of $I^{Sh}_{\Delta,r}(G,\tau)$ vanish, which implies $I^{Sh}_{\Delta,r}(G,\tau) = 0$. Conversely, the vanishing of the Shintani lift in particular means the vanishing of its Δ-th coefficient, i.e. $L(G,\chi_\Delta,k+1)^2 = 0$. This completes the proof. $\square$

To describe the Fourier coefficients of the twisted Millson lift we introduce twisted traces of CM values and cycle integrals.

For $h \in L'/L$ and $m \in \mathbb{Q}_{>0}$ with $m \equiv \operatorname{sgn}(\Delta)Q(h)$ $(\mathbb{Z})$ we define the twisted trace of a Γ-invariant function F by

$$\operatorname{tr}^+_{F,\Delta,r}(m,h) = \sum_{X \in \Gamma \backslash L^+_{|\Delta|m,rh}} \frac{\chi_\Delta(X)}{\left|\overline{\Gamma}_X\right|} F(z_X),$$

and $\operatorname{tr}^-_{F,\Delta,r}(m,h)$ accordingly.

For $h \in L'/L$ and $m \in \mathbb{Q}_{<0}$ with $m \equiv \operatorname{sgn}(\Delta)Q(h)$ $(\mathbb{Z})$ we define the twisted trace of a cusp form $G \in S_{2k+2}(\Gamma)$ by

$$\operatorname{tr}_{F,\Delta,r}(m,h) = \sum_{X \in \Gamma \backslash L_{|\Delta|m,rh}} \chi_\Delta(X)\mathcal{C}(G,X),$$

with the cycle integral $\mathcal{C}(G,X)$ defined in Section 3.3.

Finally, for $m = -N|\Delta|d^2 < 0$ with $d \in \mathbb{Q}_{>0}$ we define the twisted complementary trace by

$$\begin{aligned}\operatorname{tr}^c_{F,\Delta,r}(-N|\Delta|d^2,h) = \sum_{X \in \Gamma \backslash L_{-N|\Delta|^2 d^2,rh}} \chi_\Delta(X) \Bigg(&\sum_{w \in \mathbb{Q}_{<0}} a^+_{\ell_X}(w)(4\pi w)^k e^{2\pi i \operatorname{Re}(c(X))w} \\ &+ (-1)^{k+1} \sum_{w \in \mathbb{Q}_{<0}} a^+_{\ell_{-X}}(w)(4\pi w)^k e^{2\pi i \operatorname{Re}(c(-X))w} \Bigg).\end{aligned}$$

Theorem 3.4.3. *Let $k \in \mathbb{Z}_{\geq 0}$ and let $F \in H^{+}_{-2k}(\Gamma)$. For $k > 0$ the h-th component of $I^{M}_{\Delta,r}(F,\tau)$ is given by*

$$\begin{aligned}
&\sum_{m>0} \frac{1}{2\sqrt{m}} \left(\frac{\sqrt{N}}{4\pi\sqrt{|\Delta| m}}\right)^{k} \left(\operatorname{tr}^{+}_{R^{k}_{-2k}F,\Delta,r}(m,h) + (-1)^{k+1}\operatorname{tr}^{-}_{R^{k}_{-2k}F,\Delta,r}(m,h)\right) q^{m} \\
&\quad + \sum_{d>0} \frac{1}{2i\sqrt{N|\Delta|}d} \left(\frac{1}{4\pi i |\Delta| d}\right)^{k} \operatorname{tr}^{c}_{F,\Delta,r}(-N|\Delta|d^{2},h) q^{-N|\Delta|d^{2}} \\
&\quad + \frac{\sqrt{|\Delta|}(-1)^{k}k!}{2\sqrt{N}\pi^{k+1}} \sum_{\substack{\ell \in \Gamma\setminus \operatorname{Iso}(V) \\ \ell \cap (L+rh) \neq \emptyset}} a^{+}_{\ell}(0) \frac{\alpha_{\ell}}{\beta_{\ell}^{k+1}} d_{\ell}^{k+1} \\
&\qquad\qquad \times \left(\sum_{\substack{n>0 \\ n \equiv m_{\ell}(d_{\ell})}} \frac{\chi_{\Delta}(n)}{n^{s+1}} + (-1)^{k+1}\operatorname{sgn}(\Delta) \sum_{\substack{n>0 \\ n \equiv -m_{\ell}(d_{\ell})}} \frac{\chi_{\Delta}(n)}{n^{s+1}} \right)\Bigg|_{s=k} \\
&\quad - \sum_{m<0} \frac{1}{2(4\pi|m|)^{k+1/2}|\Delta|^{k/2}} \overline{\operatorname{tr}_{\xi_{-2k}F,\Delta,r}(m,h)} \Gamma\left(1/2+k, 4\pi|m|v\right) q^{m},
\end{aligned}$$

where $m_{\ell}, d_{\ell} \in \mathbb{Z}_{\geq 0}$ are defined by $(m_{\ell}, d_{\ell}) = 1$ and $k_{\ell}/\beta_{\ell} = m_{\ell}/d_{\ell}$.

For $k = 0$ the h-th component of $I^{M}_{\Delta,r}(F,\tau)$ is given by the same formula as above but with the additional non-holomorphic terms

$$\sum_{d>0} \frac{1}{4i\sqrt{\pi N|\Delta|}d} \sum_{X \in \Gamma\setminus L_{-N|\Delta|d^{2},rh}} \chi_{\Delta}(X)\left(a^{+}_{\ell_{X}}(0) - a^{+}_{\ell_{-X}}(0)\right)\Gamma\left(1/2, 4\pi N|\Delta|d^{2}v\right) q^{-N|\Delta|d^{2}}.$$

Proof. As in the proof of Proposition 3.4.1 we write

$$I^{M}_{\Delta,r}(F,\tau) = \frac{1}{[\Gamma : \Gamma^{(\Delta)}]} \sum_{h \in L'/L} \left\langle \psi_{\Delta,r}(\mathfrak{e}_{h}), \overline{I^{M}\left(F,\tau,L^{(\Delta)},\Gamma^{(\Delta)}\right)} \right\rangle \mathfrak{e}_{h}.$$

We see that the coefficients of the twisted lift can be obtained from the coefficients of the untwisted lift. The twisting of the coefficients of positive and negative index is quite straightforward and can be done as in the proof of Theorem 5.5. in [AE13].

We sketch the twisting of the constant coefficient. For $h \in L'/L$ with $Q(h) \equiv 0(\mathbb{Z})$

the $(0,h)-$th coefficient of $I^M_{\Delta,r}(F,\tau)$ is given by

$$\frac{\sqrt{|\Delta|}(-1)^k k!}{2\sqrt{N}\pi^{k+1}} \frac{1}{[\Gamma : \Gamma^{(\Delta)}]} \sum_{\substack{\delta \in L'/L^{(\Delta)} \\ \pi(\delta)=rh \\ Q_\Delta(\delta)\equiv 0(\mathbb{Z})}} \chi_\Delta(\delta)$$

$$\sum_{\substack{\ell \in \Gamma^{(\Delta)} \setminus \operatorname{Iso}(V) \\ \ell \cap (\Delta L+\delta)\neq\emptyset}} a_\ell^+(0) \frac{\alpha_\ell^{(\Delta)}}{\left(\beta_\ell^{(\Delta)}\right)^{k+1}} \left(\zeta\left(s, k_\ell^{(\Delta)}/\beta_\ell^{(\Delta)}\right) + (-1)^{k+1}\zeta\left(s, 1-k_\ell^{(\Delta)}/\beta_\ell^{(\Delta)}\right)\right)\Big|_{s=k+1},$$

It is easy to see that $\beta_\ell^{(\Delta)} = |\Delta|\beta_\ell$ and $\alpha_\ell^{(\Delta)} = [\Gamma_\ell : (\Gamma^{(\Delta)})_\ell]\alpha_\ell$, but $k_\ell^{(\Delta)}$ is a bit more complicated:

Let $X_\ell \in \ell \cap L$ be a positively oriented primitive generator of ℓ. If $\ell \cap (\Delta L + \delta) \neq \emptyset$ with $\pi(\delta) = rh$ then also $\ell \cap (L + rh) \neq \emptyset$. For a fixed isotropic line ℓ, a system of representatives for the elements $\delta \in L'/L^{(\Delta)}$ with $\pi(\delta) = rh, Q_\Delta(\delta) \equiv 0(\mathbb{Z})$ and $\ell \cap (\Delta L + \delta) \neq \emptyset$ is given by the vectors $nX_\ell + (rh)_\ell$ with n running modulo $|\Delta|$ and some $(rh)_\ell \in \ell \cap (L + rh)$. In particular, we have $k_\ell^{(\Delta)}/\beta_\ell^{(\Delta)} = n/|\Delta| + m_\ell/|\Delta|d_\ell$. Using the assumption that Δ is a fundamental discriminant it is not hard to show that $(\Delta, d_\ell) = 1$, $\chi_\Delta(d_\ell) = 1$, and $\chi_\Delta(nX_\ell + (rh)_\ell) = \chi_\Delta(nd_\ell + m_\ell)$. Putting everything together, we obtain the twisted constant coefficient. $\square$

In the same way, we obtain the Fourier expansion of the (Δ, r)-th Shintani lift:

Theorem 3.4.4. *Let $k \in \mathbb{Z}_{\geq 0}$ and $G \in S_{2k+2}(\Gamma)$. Then the h-th component of $I^{Sh}_{\Delta,r}(G,\tau)$ is given by*

$$I^{Sh}_{\Delta,r}(G,\tau)_h = -\frac{\sqrt{N}}{\sqrt{|\Delta|}} \sum_{m>0} \frac{1}{|\Delta|^{k/2}} \operatorname{tr}_{G,\Delta,r}(-m,h)q^m.$$

Remark 3.4.5. Let $N = 1$. In this case the twisted Millson theta function vanishes identically if $(-1)^k\Delta > 0$, which easily follows from replacing X by $-X$ in the sum. On the other hand, by Theorem 2.3.15 for $(-1)^k\Delta < 0$ the map $f_0(\tau)\mathfrak{e}_0 + f_1(\tau)\mathfrak{e}_1 \mapsto f_0(4\tau) + f_1(4\tau)$ defines an isomorphism of $H^+_{1/2-k,\tilde{\rho}_L}$ with the subspace of $H^+_{1/2-k}(\Gamma_0(4))$ of scalar valued harmonic weak Maass forms satisfying the Kohnen plus space condition. Using this identification we can derive the results stated in the introduction from the theorems in this section. Since $\Delta \equiv r^2(4)$, the value of r mod 2 is already determined by Δ, so we can drop it from the notation. The formula for the coefficients of positive index of I^M_Δ follows from $\operatorname{tr}^-_{R^k_{-2k}F,\Delta}(d) = \operatorname{sgn}(\Delta)\operatorname{tr}^+_{R^k_{-2k}F,\Delta}(d)$, and the formula for the principal part is obtained by rewriting the twisted complementary trace as described in [AE13, Proposition 5.7.].

Finally, we compute a nicer closed formula for the twisted lift of the constant function 1 by taking the residue at $s = 0$ of the lift of a non-holomorphic Eisenstein series $\mathcal{E}_0(z,s)$.

The method used here appeared in many places in the literature, for example in [BF06, Section 7.1], [AE13, Section 6.1], and [Alf15, Section 4.2], and it can also be applied to compute the lift of non-holomorphic Maass Poincaré series.

The Eisenstein series $\mathcal{E}_0(z,s)$ is given by

$$\mathcal{E}_0(z,s) = \frac{1}{2}\zeta^*(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\mathrm{Im}(\gamma z))^s,$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ denotes the completed Riemann Zeta function. Here, $\Gamma_\infty = \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$. The Eisenstein series $\mathcal{E}_0(z,s)$ converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue $\pi/(6\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H}))$.

Moreover, we define the following vector valued Eisenstein series of weight $1/2$,

$$\mathcal{E}_{1/2,\tilde{\rho}_K}(\tau,s) = \frac{1}{2} \sum_{\gamma \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} (v^{\frac{s}{2}}\mathfrak{e}_0)|_{1/2,\tilde{\rho}_K}\gamma.$$

Here, K is the sublattice $\mathbb{Z}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ of L and $\tilde{\Gamma}_\infty$ is the subgroup of $\mathrm{Mp}_2(\mathbb{Z})$ generated by $T = (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$. Its dual lattice is $K' = \frac{1}{2N}K$. Since $K'/K \cong L'/L$ we can view $\mathcal{E}_{1/2,\tilde{\rho}_K}(\tau,s)$ as a modular form for $\tilde{\rho}_L$. For $\Delta > 0$, i.e. $\tilde{\rho}_K = \rho_K$, we replace γ by $Z\gamma$ in the sum, where $Z = (-1, i) \in \mathrm{Mp}_2(\mathbb{Z})$, and use that $\mathfrak{e}_0|_{1/2,\rho_K} Z = -\mathfrak{e}_0$, to see that the Eisenstein series vanishes identically in this case.

For $\Delta < 0$ let $\chi_\Delta(n) = \left(\frac{\Delta}{n}\right)$ and let

$$\Lambda\left(\chi_\Delta, s\right) = \left(\frac{\pi}{|\Delta|}\right)^{-(s+1)/2} \Gamma((s+1)/2) \sum_{n \geq 1} \chi_\Delta(n) n^{-s}$$

be the completed Dirichlet L-series associated with χ_Δ.

Theorem 3.4.6. *For* $\mathrm{Re}(s) > 1$ *we have* $I^M_{\Delta,r}(\mathcal{E}_0(z,s),\tau) = 0$ *if* $\Delta > 0$, *and*

$$I^M_{\Delta,r}(\mathcal{E}_0(z,s),\tau) = \frac{1}{2\sqrt{|\Delta|}}\zeta^*(2s) N^{\frac{1}{2}-\frac{s}{2}} \Lambda\left(\chi_\Delta, s\right) \mathcal{E}_{1/2,\rho^*_K}(\tau,s),$$

if $\Delta < 0$.

Proof. The proof follows the one in [BF06, Theorem 7.1, Corollary 7.2] and [AE13, Theorem 6.1]. Using the standard unfolding trick we obtain

$$I^M_{\Delta,r}(\mathcal{E}_0(z,s),\tau) = \zeta^*(2s) \int_{\Gamma_\infty \backslash \mathbb{H}} \Theta_M(\tau,z) y^s d\mu(z).$$

We identify $K'/K \cong \mathbb{Z}/2N\mathbb{Z}$ with quadratic form $Q(b) = -b^2/4N \bmod \mathbb{Z}$. Then by

[Höv12, Satz 2.22] this equals

$$
\begin{aligned}
&-\zeta^*(2s)\frac{N\bar{\varepsilon}}{2\sqrt{|\Delta|}i}\sum_{n\geq 1} n\left(\frac{\Delta}{n}\right)\sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\operatorname{Mp}_2(\mathbb{Z})}\\
&\left[\frac{1}{v^{1/2}}\int_{y=0}^{\infty} y^s\exp\left(-\frac{N\pi n^2y^2}{|\Delta|\,v}\right)dy\cdot\int_{x=0}^{1}\sum_{b\in\mathbb{Z}} e\left(-|\Delta|\,b^2\bar{\tau}/4N+bnx\right)\mathfrak{e}_{rb}dx\right]\Bigg|_{1/2,\tilde{\rho}_K}\gamma,
\end{aligned}
$$

where $\varepsilon=1$ if $\Delta>0$ and $\varepsilon=i$ if $\Delta<0$. Note that, compared to the formula in [Höv12], we get an additional factor $|\Delta|^{-1/2}$ due to our different normalization of $p_z(X)$ in the twisted case. The integral over x equals $\mathfrak{e}_0$ and the one over y equals

$$
\frac{1}{2\sqrt{|\Delta|}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)(|\Delta|\,v)^{\frac{s}{2}+\frac{1}{2}}(N\pi)^{-\frac{s}{2}-\frac{1}{2}}n^{-s-1}.
$$

Thus, we have

$$
\begin{aligned}
I^M_{\Delta,r}(\mathcal{E}_0(z,s),\tau)=&-\zeta^*(2s)N^{-\frac{s}{2}+\frac{1}{2}}\Gamma\left(\frac{s+1}{2}\right)\frac{\bar{\varepsilon}}{2\sqrt{|\Delta|}i}|\Delta|^{\frac{s+1}{2}}\pi^{-\frac{s+1}{2}}\\
&\times\left(\sum_{n\geq 1}\left(\frac{\Delta}{n}\right)n^{-s}\right)\left(\frac{1}{2}\sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\operatorname{Mp}_2(\mathbb{Z})}(v^{\frac{s}{2}}\mathfrak{e}_0)|_{1/2,\tilde{\rho}_K}\gamma\right).
\end{aligned}
$$

For $\Delta>0$ the Eisenstein series vanishes identically, and for $\Delta<0$ we plug in $\varepsilon=i$ and find

$$
I^M_{\Delta,r}(\mathcal{E}_0(z,s),\tau)=\frac{1}{2\sqrt{|\Delta|}}\zeta^*(2s)N^{\frac{1}{2}-\frac{s}{2}}\Lambda\left(\chi_\Delta,s\right)\mathcal{E}_{1/2,\overline{\rho}_K}(\tau,s).
$$

This completes the proof. □

We now take residues at $s=1$ in Theorem 3.4.6 to compute the lift of the constant function.

Lemma 3.4.7. *The residue of $\mathcal{E}_{1/2,\rho_K^*}(\tau,s)$ at $s=1$ is given by*

$$
\frac{2}{\operatorname{vol}(\Gamma_0(N)\backslash\mathbb{H})}\sum_{\ell\in\Gamma_0(N)\backslash\operatorname{Iso}(V)}\frac{\varepsilon_\ell}{\sqrt{N}}\overline{\Theta_{\ell,0}(\tau)}\in M_{1/2,\rho_L^*}.
$$

Proof. Repeating the arguments in [BFI15, Section 5.5.1] for higher level N, we see that the residue of $N^{\frac{1}{2}-\frac{s}{2}}\frac{\pi\zeta^*(s)}{12\zeta^*(2s-1)}\mathcal{E}_{1/2,\rho_K^*}(\tau,s)$ at $s=1$ is given by

$$\sum_{\ell\in\Gamma_0(N)\backslash\mathrm{Iso}(V)}\frac{B_\ell(1)\varepsilon_\ell}{\sqrt{N}}\overline{\Theta_{\ell,0}(\tau)},$$

where $B_\ell(s)$ is the function appearing in the constant coefficient $A_\ell(s)y^s+B_\ell(s)y^{1-s}$ of the Fourier expansion of $\frac{\zeta^*(2)}{\zeta^*(2s)\zeta^*(2s-1)}\mathcal{E}_0(\tau,s)$ at the cusp ℓ. Let us write $\tilde{A}_\ell(s)y^s+\tilde{B}_\ell(s)y^{1-s}$ for the constant coefficient of $\mathcal{E}_0(\tau,s)$ at the cusp ℓ. Using the well known fact that $\tilde{B}_\ell(s)$ has residue $\pi/(6\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H}))$ at $s=1$, independently of ℓ, and the expansion $\frac{\zeta^*(2)}{\zeta^*(2s)\zeta^*(2s-1)}=2(s-1)+O((s-1)^2)$, we obtain

$$B_\ell(1)=\frac{\pi}{3\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H})}.$$

Finally, using $\left(\frac{\zeta^*(s)}{\zeta^*(2s-1)}\right)|_{s=1}=2$, we get the stated formula. □

Proposition 3.4.8. *For $\Delta>0$ we have $I^M_{\Delta,r}(1,\tau)=0$, and for $\Delta<0$ we have*

$$I^M_{\Delta,r}(1,\tau)=\frac{\Lambda(\chi_\Delta,1)}{\sqrt{N|\Delta|}}\sum_{\ell\in\Gamma_0(N)\backslash\,\mathrm{Iso}(V)}\varepsilon_\ell\overline{\Theta_{\ell,0}(\tau)}.$$

In both cases we have $I^M_{\Delta,r}(1,\tau)\in M_{1/2,\tilde{\rho}_L}$.

3.5 Extensions of the Millson and the Shintani theta lift

The Millson and the Shintani theta lifts can be generalized to the full spaces of harmonic Maass forms $H_{-2k}(\Gamma)$ and $H_{2k+2}(\Gamma)$, respectively. Since the necessary computations become even more technical, we only sketch the arguments. Further, for simplicity we restrict to the space $H^0_{-2k}(\Gamma)$ consisting of harmonic Maass forms whose coefficients $a_\ell^-(0)$ in the non-holomorphic parts of the input at all cusps vanish. The extension of the Shintani lift to harmonic Maass forms will be the topic of upcoming joint work with Alfes-Neumann [ANS17].

Concerning the regularization, the proof of Proposition 3.1.1 still goes through, so the Millson lift of $F\in H^0_{-2k}(\Gamma)$ converges to a harmonic function transforming like a modular form of weight $1/2-k$ for ρ_L. Similarly, the Shintani lift of a harmonic Maass forms $G\in H^0_{2k+2}(\Gamma)$ can be regularized as in [BFI15], Definition 5.6, and yields a harmonic function transforming of weight $3/2+k$ for ρ_L^*. The relation between the Millson and the Shintani theta lift given in Proposition 3.1.2 still holds for $F\in H^0_{-2k}(\Gamma)$, which can be seen by exactly the same proof as for $F\in H^+_{-2k}(\Gamma)$. Additionally, we have the 'converse' relation

$$\xi_{3/2+k,\tau} I^{Sh}(G,\tau) = -\frac{\sqrt{N}}{2} I^M(\xi_{2k+2,z}G,\tau).$$

In particular, this means that the Shintani lift of a harmonic Maass form $G \in H^0_{2k+2}(\Gamma)$ produces a $\xi_{3/2+k}$-preimage of the Millson lift of $\xi_{2k+2}G$. For example, for $N = 1$ and $k = 0$ one can use it to construct $\xi_{3/2}$-preimages of the weakly holomorphic modular forms $f_d = q^{-d} + O(q) \in M^{!,+}_{1/2}(\Gamma_0(4))$ by applying twisted Shintani lifts to a ξ_2-preimage of $J = j - 744$.

The computation of the Fourier expansion of $I^M(F,\tau)$ for $F \in H^0_{-2k}(\Gamma)$ can be done in a similar way as before, but we have to be careful with the main integral in the computation of the coefficients of negative index since $\xi_{-2k}F$ need no longer be a cusp form. However, it might be better to adapt the more conceptual method of [BFI15], which in our case relies on the existence of a Δ_k-preimage of the Millson Schwartz function and Stokes' theorem, as we now briefly explain. Using the rules (2.4.2), a short calculation shows that for $Q_X(z) \neq 0$ the function

$$\mu(X,\tau,z) = -\frac{Q_X^k(z)\,\mathrm{sgn}(p_X(z))}{\sqrt{2}(2\pi)^{k+1}} \frac{\sqrt{\pi}}{2} \int_{2\pi v}^{\infty} e^{2Q(X)w} w^{k+1/2}\,\mathrm{erfc}\left(|p_X(z)|\sqrt{w/2}\right)\frac{dw}{w}$$

satisfies

$$\xi_{-2k,z}\mu(X,\tau,z) = \frac{N^{k+1/2}}{2\pi^{k+1}Q_X^{k+1}(z)}\Gamma\left(k+1, 2\pi v R(X,z)\right)$$

and

$$\Delta_{-2k,z}\mu(X,\tau,z) = \overline{\psi_M^0(X,\tau,z)}.$$

Note that $\xi_{-2k,z}\mu(X,\tau,z)$ equals $-\eta(X,\tau,z)$, which is the function used for the calculation of the Fourier coefficients of the Millson lift of negative index. In the computation of the Fourier coefficients, we can now essentially just shift Δ_k to F in the theta integral by Stokes' theorem (see Lemma 2.3.9), leaving us with two boundary integrals. They are still difficult to compute, but can be handled in a similar way as in [BFI15], Section 8. However, $\mu(X,\tau,z)$ has a jump singularity along the geodesic c_X if $Q(X) < 0$, and $\xi_{-2k,z}\mu(X,\tau,z)$ has a pole of order $k+1$ at the CM point z_X if $Q(X) > 0$, so in order to apply Stokes' theorem we first have to cut out small neighbourhoods around these singularities from the truncated curve M_T. Thus we obtain additional integrals along the boundaries of these neighbourhoods, which yield the traces of CM values and (regularized) cycle integrals appearing in the Fourier expansion of the lift. For the Shintani lift, one can find a similar Δ_k-preimage which makes it possible to generalize the lift to the full space $H_{2k+2}(\Gamma)$, and which will be discussed in [ANS17].

The twisting of these extended lifts proceeds in the same way as before. We obtain the following extension of (the twisted versions of) Proposition 3.1.2 and Theorem 3.1.3.

Theorem 3.5.1. *Let $k \in \mathbb{Z}_{\geq 0}$.*

1. *The Millson theta lift $I^M_{\Delta,r}$ maps $H^0_{-2k}(\Gamma)$ to $H_{1/2-k,\tilde{\rho}_L}$.*
2. *The Shintani theta lift $I^{Sh}_{\Delta,r}$ maps $H^0_{2k+2}(\Gamma)$ to $H_{3/2+k,\tilde{\rho}_L^*}$.*
3. *The relation between the Millson and the Shintani theta lift in Proposition 3.4.1 also holds for $F \in H^0_{-2k}(\Gamma)$.*

3.6 Cycle integrals

We have seen above that the Fourier coefficients of negative index of the non-holomorphic part of the Millson lift of a harmonic Maass form $F \in H^+_{-2k}(\Gamma)$ are given by traces of cycle integrals of $\xi_{-2k}F$. By Theorem 3.2.3 the Millson lift $I^M(F,\tau)$ agrees (up to some constant factor) with the lift $\Lambda^M(F,\tau)$ constructed from the weight 0 Millson theta function and iterated lowering and raising operators. It is possible to compute the Fourier coefficients of negative index of $\Lambda^M(F,\tau)$ in a similar way as we did for $I^M(F,\tau)$ above. It turns out that they are given by traces of cycle integrals of $R^{k+1}_{-2k}F$. We will not give these calculations here, but, inspired by this observation, we will prove some interesting identities between the cycle integrals of $\xi_{-2k}F$ and $R^{2j+1}_{-2k}F$, $j \geq 0$, in a more direct way.

3.6.1 Closed geodesics

Let $X \in V$ with $Q(X) = m < 0$ such that $|m|/N$ is not a square in $\mathbb{Q}$, i.e.,the stabilizer $\overline{\Gamma}_X$ is infinite cyclic and $c(X) = \Gamma_X \backslash c_X$ is a closed geodesic. Further, let G be some smooth function that transforms like a modular form of weight $2k+2$ under Γ for some $k \in \mathbb{Z}$. Recall the definition of the cycle integral

$$\mathcal{C}(G,X) = (-2\sqrt{|m|N}i)^k i \int_1^{\varepsilon^2} G_g(iy)y^k dy,$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ is such that $g^{-1}Xg = \sqrt{|m|/N}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\varepsilon > 1$ is such that $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$ generates the stabilizer of $g^{-1}Xg$ in $g^{-1}\overline{\Gamma}g$, and $G_g = G|_{2k+2}g$.

Proposition 3.6.1. *Let $X \in V$ with $Q(X) = m < 0$ such that $|m|/N$ is not a square. Let $k \in \mathbb{Z}$ and $F \in H^+_{-2k}(\Gamma)$. For all integers $\ell \leq k$ we have*

$$\mathcal{C}(R^{k-\ell+1}_{-2k}F, X) = \frac{1}{(4|m|N)^\ell}\overline{\mathcal{C}(\xi_{-2\ell}R^{k-\ell}_{-2k}F, X)}. \tag{3.6.1}$$

Further, for $\ell \leq k-1$ we have

$$\mathcal{C}(R^{k-\ell+1}_{-2k}F, X) = 4|m|N(k-\ell)(k+\ell+1)\mathcal{C}(R^{k-\ell-1}_{-2k}F, X). \tag{3.6.2}$$

Proof. Plugging in the definition of the cycle integral, the left-hand side of (3.6.1) equals

$$(-2\sqrt{|m|N}i)^{-\ell}i\int_1^{\varepsilon^2}(R_{-2k}^{k-\ell+1}F_g)(iy)y^{-\ell}dy.$$

Since $\ell\leq k$ we can split off the outermost raising operator $R_{-2\ell}=2i\frac{\partial}{\partial z}-2\ell y^{-1}$ to obtain

$$(R_{-2k}^{k-\ell+1}F_g)(iy)y^{-\ell}=2i\left(\frac{\partial}{\partial z}R_{-2k}^{k-\ell}F_g\right)(iy)y^{-\ell}-2\ell(R_{-2k}^{k-\ell}F_g)(iy)y^{-\ell-1}.$$

Now we use $\frac{\partial}{\partial z}=\frac{\partial}{\partial \bar{z}}-i\frac{\partial}{\partial y}$ and apply the product rule to the $\frac{\partial}{\partial y}$-part to get

$$(R_{-2k}^{k-\ell+1}F_g)(iy)y^{-\ell}=2i\left(\frac{\partial}{\partial \bar{z}}R_{-2k}^{k-\ell}F_g\right)(iy)y^{-\ell}+2\frac{\partial}{\partial y}\left((R_{-2k}^{k-\ell}F_g)(iy)y^{-\ell}\right).$$

Note that we also used $(\frac{\partial}{\partial y}R_{-2k}^{k-\ell}F_g)(iy)=\frac{\partial}{\partial y}((R_{-2k}^{k-\ell}F_g)(iy))$. The first summand on the right-hand side equals

$$2i\left(\frac{\partial}{\partial \bar{z}}R_{-2k}^{k-\ell}F_g\right)(iy)y^{-\ell}=-\overline{(\xi_{-2\ell}R_{-2k}^{k-\ell}F_g)(iy)}y^{\ell},$$

giving the right-hand side of (3.6.1). Further, the integral

$$\int_1^{\varepsilon^2}\frac{\partial}{\partial y}\left((R_{-2k}^{k-\ell}F_g)(iy)y^{-\ell}\right)dy=(R_{-2k}^{k-\ell}F_g)(i\varepsilon^2)\varepsilon^{-2\ell}-(R_{-2k}^{k-\ell}F_g)(i)$$

vanishes since $(R_{-2k}^{k-\ell}F_g)(i\varepsilon^2)\varepsilon^{-2\ell}=(R_{-2k}^{k-\ell}F_g)|_{-2\ell}\left(\begin{smallmatrix}\varepsilon&0\\0&\varepsilon^{-1}\end{smallmatrix}\right)(i)$ and $R_{-2k}^{k-\ell}F_g$ transforms like a modular form of weight -2ℓ for $g^{-1}\Gamma g$. This completes the proof of (3.6.1).

The formula (3.6.2) easily follows from (3.6.1) if we use that

$$\overline{\xi_{-2\ell}R_{-2k}^{k-\ell}F}=(k-\ell)(k+\ell+1)y^{-2\ell-2}R_{-2k}^{k-\ell-1}F$$

for all $k\in\mathbb{Z}$, all integers $\ell\leq k-1$ and $F\in H_{-2k}^{+}(\Gamma)$. This follows from Lemma 2.3.5 if we write $\xi_{-2\ell}=y^{-2\ell-2}\overline{L_{-2\ell}}$ and use the relation (2.3.3). □

Corollary 3.6.2. *Let $X\in V$ with $Q(X)=m<0$ such that $|m|/N$ is not a square. Further, let $k\in\mathbb{Z}_{\geq0}$ and $F\in H_{-2k}^{+}(\Gamma)$. For $j\in\mathbb{Z}_{\geq0}$ we have*

$$\mathcal{C}(R_{-2k}^{2j+1}F,X)=\frac{1}{(4|m|N)^{k-j}}\frac{j!(k-j)!(2k)!}{k!(2k-2j)!}\overline{\mathcal{C}(\xi_{-2k}F,X)}.$$

Proof. We use (3.6.1) with $\ell=k$ and then repeatedly apply (3.6.2). □

As a we special case we obtain a generalization of Theorem 1.1. from [BGK14].

Corollary 3.6.3. *Let $X \in V$ with $Q(X) = m < 0$ such that $|m|/N$ is not a square. Further, let $k \in \mathbb{Z}_{\geq 0}$ and $F \in H^+_{-2k}(\Gamma)$. For even k we have*

$$\mathcal{C}(R^{k+1}_{-2k}F, X) = \frac{1}{(4|m|N)^{k/2}} \frac{((\frac{k}{2})!)^2(2k)!}{(k!)^2} \overline{\mathcal{C}(\xi_{-2k}F, X)},$$

and for odd k we have

$$\mathcal{C}(R^{k}_{-2k}F, X) = \frac{1}{(4|m|N)^{(k+1)/2}} \frac{(\frac{k-1}{2})!(\frac{k+1}{2})!(2k)!}{(k+1)!k!} \overline{\mathcal{C}(\xi_{-2k}F, X)}.$$

Moreover, we obtain the non-square part of Theorem 1.1 from [BGK15] which asserts that the cycle integrals of the weight $2k+2$ weakly holomorphic modular forms

$$D^{2k+1}F = -(4\pi)^{-(2k+1)}R^{2k+1}_{-2k}F$$

and $\xi_{-2k}F$ agree up to some constant.

Corollary 3.6.4. *Let $X \in V$ with $Q(X) = m < 0$ such that $|m|/N$ is not a square. For $k \in \mathbb{Z}_{\geq 0}$ and $F \in H^+_{-2k}(\Gamma)$ we have*

$$\mathcal{C}(D^{2k+1}F, X) = -\frac{(2k)!}{(4\pi)^{2k+1}} \overline{\mathcal{C}(\xi_{-2k}F, X)}.$$

3.6.2 Infinite geodesics

Let $X \in V$ with $Q(X) = m < 0$ such that $|m|/N$ is a square in $\mathbb{Q}$, i.e. the stabilizer $\overline{\Gamma}_X$ is trivial and $c(X) = \Gamma_X \backslash c_X$ is an infinite geodesic in $\Gamma \backslash \mathbb{H}$. Recall that for a cusp form $G \in S_{2k+2}$ the cycle integral is defined by

$$\mathcal{C}(G, X) = (-2\sqrt{|m|N}i)^k i \int_0^\infty G_g(iy)y^k dy,$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ is such that $g^{-1}Xg = \sqrt{|m|/N}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $G_g = G|_{2k+2}g$.

We would like to prove similar identities as in the last section, but in general the cycle integral of $R^{k-\ell}_{-2k}F$ does not converge if the geodesic is infinite. If we start with the (convergent) cycle integral of $\xi_{-2k}F$ and repeat the calculations of the last section, we are led to suitable regularized cycle integrals of $R^{k-\ell}_{-2k}F$.

First, for $F \in H^+_{-2k}(\Gamma)$ we write

$$\mathcal{C}(\xi_{-2k}F, X) = (-2\sqrt{|m|N}i)^k i \left(\int_1^\infty \xi_{-2k}F_g(iy)y^k dy + (-1)^{k+1} \int_1^\infty \xi_{-2k}F_{gS}(iy)y^k dy \right),$$

with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, where we split the integral over $(0, \infty)$ at 1 and replaced y by $1/y$ in the integral over $(0,1)$. Now we decompose $F_g = F_g^+ + F_g^-$ and $F_{gS} = F_{gS}^+ + F_{gS}^-$ into their holomorphic and non-holomorphic parts and use $\xi_{-2k}F_g = \xi_{-2k}F_g^-$ and $\xi_{-2k}F_{gS} = \xi_{-2k}F_{gS}^-$. Note that F_g^- and F_{gS}^- are rapidly decreasing at the cusp ∞, but not necessarily at 0, and this is the reason why we split the integral above. We have the following analog of Proposition 3.6.1:

Proposition 3.6.5. *Let $X \in V$ with $Q(X) = m < 0$ such that $|m|/N$ is a square. Let $k \in \mathbb{Z}$ and $F \in H_{-2k}^+(\Gamma)$. For all integers $\ell \leq k$ we have*

$$\int_1^\infty R_{-2k}^{k-\ell+1} F_g^-(iy) y^{-\ell} dy = -\overline{\int_1^\infty \xi_{-2\ell} R_{-2k}^{k-\ell} F_g^-(iy) y^{\ell} dy} - 2R_{-2k}^{k-\ell} F_g^-(i).$$

Further, for $\ell \leq k-1$ we have

$$\int_1^\infty R_{-2k}^{k-\ell+1} F_g^-(iy) y^{-\ell} dy$$
$$= -(k-\ell)(k+\ell+1)\int_1^\infty R_{-2k}^{k-\ell-1} F_g^-(iy) y^{-\ell-2} dy - 2R_{-2k}^{k-\ell} F_g^-(i).$$

The same formulas hold with g replaced by gS.

Proof. The computations are the same as in the proof of Proposition 3.6.1 if we replace ε^2 by ∞ and use the rapid decay of $R_{-2k}^{k-\ell} F^-(iy)$ as $y \to \infty$. □

A repeated application of the proposition leads to following definition: For every integer $j \geq 0$ we define the regularized cycle integral of $R_{-2k}^{2j+1} F$ by

$$\begin{aligned} \mathcal{C}^{\mathrm{reg}}(R_{-2k}^{2j+1}F, X) &= (-2\sqrt{|m|N}i)^{-k+2j} i \\ &\times \Bigg(\sum_{\ell=0}^{j} C_{\ell,j}(R_{-2k}^{2\ell} F_g^-(i)) + (-1)^{k+1} \sum_{\ell=0}^{j} C_{\ell,j}(R_{-2k}^{2\ell} F_{gS}^-(i)) \\ &\quad + \int_1^\infty R_{-2k}^{2j+1} F_g^-(iy) y^{-k+2j} dy + (-1)^{k+1} \int_1^\infty R_{-2k}^{2j+1} F_{gS}^-(iy) y^{-k+2j} dy \Bigg), \end{aligned}$$

where $C_{\ell,j} = 2(-1)^{\ell+j} \prod_{t=\ell+1}^{j} (2t)(2k-2t+1)$. Note that

$$R_{-2k}^{2\ell} F_g^-(i) + (-1)^{k+1} R_{-2k}^{2\ell} F_{gS}^-(i) = -R_{-2k}^{2\ell} F_g^+(i) - (-1)^{k+1} R_{-2k}^{2\ell} F_{gS}^+(i),$$

so the second line above can also be understood as the part of the regularized cycle integral coming from F^+.

With this definition, we find

$$\mathcal{C}^{\mathrm{reg}}(R_{-2k}F, X) = \frac{1}{(4|m|N)^k} \overline{\mathcal{C}(\xi_{-2k}F, X)}$$

and

$$\mathcal{C}^{\mathrm{reg}}(R_{-2k}^{2j+1}F, X) = 4|m|N(2j)(2k-2j+1)\mathcal{C}^{\mathrm{reg}}(R_{-2k}^{2j-1}F, X)$$

for $j \geq 1$, and thus all the corollaries of the last section also hold for $|m|/N$ being a square.

Remark 3.6.6. For $k = j = 0$ and $F \in H_0^+(\Gamma)$ the regularized cycle integral of R_0F is defined by

$$\mathcal{C}^{\mathrm{reg}}(R_0F, X) = 2iF_g^-(i) - 2iF_{gS}^-(i) + i\int_1^\infty R_0F_g^-(iy)dy - \int_1^\infty R_0F_{gS}(iy)dy.$$

On the other hand, since $R_0F \in S_2^!(\Gamma)$ is in fact a weakly holomorphic cusp form, there is a regularized cycle integral studied in [BFK14],[BGK14] and [BGK15]. It is given by

$$\mathcal{C}_{\mathrm{BFK}}^{\mathrm{reg}}(R_0F, X) = \left[i\int_1^\infty R_0F_g(iy)e^{-ys}dy\right]\Bigg|_{s=0} - \left[i\int_1^\infty R_0F_{gS}(iy)e^{-ys}dy\right]\Bigg|_{s=0},$$

where the expression on the right means that one has to take the value at $s = 0$ of the analytic continuation of the integral. We want to compare the two regularizations. Let us split $F_g = F_g^+ + F_g^-$. Due to the rapid decay of F_g^-, we can plug in $s = 0$ in the integral over F_g^-. In the integral over F_g^+, we insert the Fourier expansion $F_g^+(z) = \sum_n a_g^+(n)e^{2\pi i n z}$, apply $R_0 = 2i\frac{\partial}{\partial z}$ and obtain after a short calculation

$$\begin{aligned}&\left[i\int_1^\infty R_0F_g^+(iy)e^{-ys}dy\right]\Bigg|_{s=0}\\ &\qquad = \left[-4\pi i\sum_{n\neq 0}\frac{na_g^+(n)}{2\pi n+s}e^{-(2\pi n+s)}\right]\Bigg|_{s=0} = -2iF_g^+(i) + 2ia_g^+(0).\end{aligned}$$

Using $F_g^+(i) - F_{gS}^+(i) = -F_g^-(i) + F_{gS}^-(i)$ we find

$$\mathcal{C}^{\mathrm{reg}}(R_0F, X) = \mathcal{C}_{\mathrm{BFK}}^{\mathrm{reg}}(R_0F, X) - 2ia_g^+(0) + 2ia_{gS}^+(0).$$

Note that the regularized cycle integrals considered in [BFK14] are only studied for weakly holomorphic cusp forms, and the analytic continuation of the integrals relies on the particular shape of the Fourier expansion of such forms. For general k and j, the function $R_{-2k}^{2j+1}F$ is not weakly holomorphic and has a somewhat complicated Fourier expansion, so it is not clear that the regularization of [BFK14] works. It would be interesting to investigate this problem in the future.

Finally, we remark that our regularized cycle integrals look very similar to the cycle integrals of weight zero harmonic weak Maass forms given in [BFI15]. However, the definitions do not overlap since we only consider cycle integrals of $R_{-2k}^{\ell}F$ for odd ℓ.

4 Applications of the Millson and the Kudla-Millson Theta Lifts

Throughout this chapter, we let L be the lattice related to $\Gamma_0(N)$ from Section 2.2.5. Recall that $L'/L \cong \mathbb{Z}/2N\mathbb{Z}$ with the quadratic form $x \mapsto -x^2/4N$. To simlify the notation, we will not distinguish between elements $h \in L'/L$ and residue classes in $\mathbb{Z}/2N\mathbb{Z}$. Further, the norms $Q(X)$ for $X \in L'$ have the form $-D/4N$ where $D \in \mathbb{Z}$ is a discriminant which is a square mod $4N$, and the elements of L' of norm $Q(X) = -D/4N$ correspond to integral binary quadratic forms Q_X of discriminant D.

4.1 Algebraic formulas for Ramanujan's mock theta functions

As an application of the Millson theta lift (for $k = 0$), we find finite algebraic formulas for the coefficients of Ramanujan's third order mock theta functions

$$\begin{aligned} f(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} \\ &= 1 + q - 2q^2 + 3q^3 - 3q^4 - 5q^5 + 7q^6 - 6q^7 + \ldots \end{aligned}$$

and

$$\begin{aligned} \omega(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2 \cdots (1-q^{2n+1})^2} \\ &= 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + 10q^6 + 14q^7 + \ldots \end{aligned}$$

in terms of the traces of a single modular function. We obtain the following result.

Theorem 4.1.1. *Consider the $\Gamma_0(6)$-invariant weakly holomorphic modular function*

$$F(z) = -\frac{1}{40} \cdot \frac{E_4(z) + 4E_4(2z) - 9E_4(3z) - 36E_4(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} \tag{4.1.1}$$

$$= q^{-1} - 4 - 83q - 296q^2 + \ldots, \tag{4.1.2}$$

where E_4 denotes the normalized Eisenstein series of weight 4 for $\mathrm{SL}_2(\mathbb{Z})$ and $\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function.

1. *For $n \geq 1$ the coefficients $a_f(n)$ of $f(q)$ are given by*

$$a_f(n) = -\frac{1}{\sqrt{24n-1}} \operatorname{Im}\left(\operatorname{tr}_F^+\left(\frac{1-24n}{24}, 1\right)\right).$$

2. *For $n \geq 1$ the coefficients $a_\omega(n)$ of $\omega(q)$ are given by*

$$a_\omega(n) = \begin{cases} \dfrac{1}{4\sqrt{24\frac{n+1}{2}-4}} \operatorname{Im}\left(\operatorname{tr}_F^+\left(\dfrac{4-24\frac{n+1}{2}}{24}, 2\right)\right), & n \text{ odd},\\[2ex] \dfrac{1}{4\sqrt{24(\frac{n}{2}+1)-16}} \operatorname{Im}\left(\operatorname{tr}_F^+\left(\dfrac{16-24\left(\frac{n}{2}+1\right)}{24}, 4\right)\right), & n \text{ even}. \end{cases}$$

Remark 4.1.2.
1. The theorem extends results of Alfes-Neumann (see the example after Theorem 1.3 in [Alf14]), who gave similar formulas for the coefficients $a_f(n)$ with $1-24n$ being a fundamental discriminant, by looking at the Kudla-Millson theta lift of F and employing a duality result between the Millson and the Kudla-Millson lift.

2. Using the Kudla-Millson theta lift, Ahlgren and Andersen [AA16] gave a formula for the smallest parts function in terms of traces of a modular function. The coefficients of Ramanujan's mock theta functions are related to partitions as well. For example, $a_f(n)$ is the number of partitions of n with even rank minus the number with odd rank, where the rank of a partition is its largest part minus the number of parts.

3. One of the main ingredients in the proof is Zwegers' [Zwe01] realization of the mock theta functions $f(q)$ and $\omega(q)$ as the holomorphic parts of the components of a vector valued harmonic Maass form. Thus the same idea works for other mock theta functions as well, for example for the order 5 and order 7 mock theta functions treated in Zwegers' thesis [Zwe02]. These cases have been treated very recently by Jennifer Kupka in her Master's thesis [Kup17].

4. We checked the above formulas numerically using Sage [Dev11].

Example 4.1.3. We illustrate our formulas by computing $a_\omega(1) = 2$. A system of representatives of the $\Gamma_0(6)$-classes of positive definite forms $ax^2+bxy+cy^2$ with $6 \mid c$, $b \equiv 2(12)$ and discriminant -20 is given by the two forms

$$Q_1 = 5x^2+10xy+6y^2, \qquad Q_2 = 7x^2+34xy+42y^2,$$

and the corresponding CM-points are

$$z_{Q_1} = \frac{10 + i\sqrt{20}}{12}, \qquad z_{Q_2} = \frac{34 + i\sqrt{20}}{84}.$$

Plugging these values into the Fourier expansion of F, we find (using Sage [Dev11])

$$F(z_{Q_1}) = F(z_{Q_2}) = i \cdot 17.888543820000.$$

Thus we obtain

$$a_\omega(1) = \frac{1}{4\sqrt{20}} \cdot 2 \cdot 17.888543820000 = 2.000000000000.$$

Proof of Theorem 4.1.1. Zwegers [Zwe01] showed that the function

$$(q^{-\frac{1}{24}} f(q), 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}), 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T$$

is the holomorphic part of a vector valued harmonic Maass form $H = (h_0, h_1, h_2)^T$, transforming as

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau), \quad H\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau).$$

Further, $\xi_{1/2} H$ is a vector consisting of cuspidal unary theta functions of weight $3/2$.

One can check that

$$\tilde{H} = (0, h_0, h_2 - h_1, 0, -h_1 - h_2, -h_0, 0, h_0, h_1 + h_2, 0, h_1 - h_2, -h_0)^T$$

is a vector valued harmonic Maass form of weight $1/2$ for the Weil representation ρ_L, compare [BO10b], Lemma 8.1. We see that its principal part is given by

$$q^{-1/24}(\mathfrak{e}_1 - \mathfrak{e}_5 + \mathfrak{e}_7 - \mathfrak{e}_{11}).$$

The function

$$(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2$$

in the denominator of F is a cusp form of weight 4 for $\Gamma_0(6)$ which is invariant under all Atkin-Lehner involutions W_d^6 for $d \mid 6$, and the numerator of F equals

$$E_4 | (W_1^6 + W_2^6 - W_3^6 - W_6^6).$$

Thus F is an eigenfunction of all Atkin-Lehner involutions, with eigenvalue 1 for W_1^6 and W_2^6, and eigenvalue -1 for W_3^6 and W_6^6. In particular, the Fourier expansions of F at the cusps of $\Gamma_0(6)$ are essentially the same, up to a possible minus sign, and the constant coefficients of F at all cusps vanish. Using the formula for the Fourier expansion of the Millson lift given in Theorem 3.3.1, it is now straightforward to check that F lifts to a harmonic Maass form $I^M(F,\tau)$ of weight $1/2$ for ρ_L, whose principal part equals $-2\sqrt{6}i$ times the principal part of $\tilde{H}$. In view of Lemma 2.3.6, this implies that the difference $\tilde{H} - \frac{i}{2\sqrt{6}} I^M(F,\tau)$ is a cusp form. But $S_{1/2,\rho_L} \cong J_{1,6}^{\text{cusp}} = \{0\}$, so we find

$$\tilde{H} = \frac{i}{2\sqrt{6}} I^M(F,\tau).$$

The holomorphic coefficients of $\frac{i}{2\sqrt{6}} I^M(F,\tau)$ at $q^{(24n-h^2)/24}\mathfrak{e}_h$ for $h^2 - 24n < 0$ are given by

$$\begin{aligned} \frac{i}{2\sqrt{24n-h^2}} &\left(\operatorname{tr}_F^+\left(\frac{h^2-24n}{24}, h\right) - \operatorname{tr}_F^-\left(\frac{h^2-24n}{24}, h\right)\right) \\ &= -\frac{1}{\sqrt{24n-h^2}} \operatorname{Im}\left(\operatorname{tr}_F^+\left(\frac{h^2-24n}{24}, h\right)\right), \end{aligned}$$

where we used that F has real coefficients and hence $\operatorname{tr}_F^+(m,h) = \overline{\operatorname{tr}_F^-(m,h)}$. Comparing the holomorphic parts of $\tilde{H}$ and $\frac{i}{2\sqrt{6}} I^M(F,\tau)$, we obtain the stated formulas for the coefficients $a_f(n)$ and $a_\omega(n)$. □

4.2 ξ-preimages of unary theta functions and rationality results

In [BFO09] and [BO10a], Bringmann, Folsom and Ono constructed scalar valued harmonic Maass forms of weight $3/2$ and $1/2$ whose shadows are the components of the unary theta functions $\theta_{1/2}$ and $\theta_{3/2}$ defined in Section 2.3.6. In both cases, the proof of the modularity of their ξ-preimages relies on transformation properties of various hypergeometric functions and q-series. Here we construct ξ-preimages for both $\theta_{1/2}$ and $\theta_{3/2}$ using the Kudla-Millson and the Millson theta lift of a single weakly holomorphic modular function F for $\Gamma_0(N)$. A nice feature of this approach is that the modularity is clear from the construction. Further, the coefficients of the holomorphic parts of these harmonic Maass forms are given by modular traces of F, and thus have good arithmetic properties. Therefore, we obtain rationality results for the holomorphic parts of harmonic Maass forms which map to the space of unary theta functions under ξ.

Let $\mathbb{C}((q))$ be the ring of formal Laurent series and let $\mathbb{C}[[q]]$ be the ring of formal power series in q. If $f = \sum_n a(n)q^n \in \mathbb{C}((q))$, we call the polynomial

$$P_f = \sum_{n\leq 0} a(n)q^n \in \mathbb{C}[q^{-1}]$$

the **principal part** of f. There is a bilinear pairing

$$\mathbb{C}((q)) \times \mathbb{C}[[q]] \to \mathbb{C}, \quad (f,g) \mapsto \{f,g\} := \text{coefficient of } q^0 \text{ of } f \cdot g.$$

It only depends on the principal part of f.

Let $k > 0$ be an even integer. We denote by $M_k^{!,\infty}(\Gamma_0(N)) \subset M_k^!(\Gamma_0(N))$ the subspace of those weakly holomorphic modular forms which vanish at all cusps different from ∞. We view the space $M_{2-k}^{!,\infty}(\Gamma_0(N))$ as a subspace of $\mathbb{C}((q))$ and view $M_k(\Gamma_0(N))$ as a subspace of $\mathbb{C}[[q]]$ by taking q-expansions at the cusp ∞.

Lemma 4.2.1. *Let $P \in \mathbb{C}[q^{-1}]$. There exists an $F \in M_{2-k}^{!,\infty}(\Gamma_0(N))$ with prescribed principal part $P_F = P$ at the cusp ∞, if and only if $\{P,g\} = 0$ for all $g \in M_k(\Gamma_0(N))$.*

This can be proved by varying the argument of [Bor99], Theorem 3.1. By Serre duality it can be shown that the subspace $M_{2-k}^{!,\infty}(\Gamma_0(N)) \subset \mathbb{C}((q))$ is the orthogonal complement of $M_k(\Gamma_0(N)) \subset \mathbb{C}[[q]]$ with respect to the pairing $\{\cdot,\cdot\}$.

We use this lemma to construct a suitable input F for the two theta lifts.

Lemma 4.2.2. *Let $k \in \mathbb{Z}_{>0}$ be even. There exists $\Gamma_0(N)$-invariant weakly holomorphic modular form*

$$F(z) = \sum_{n \gg -\infty} a(n) q^n \in M_{2-k}^{!,\infty}(\Gamma_0(N))$$

with the following properties:

1. *The Fourier coefficients $a(n)$ of F at ∞ lie in $\mathbb{Q}$.*
2. *The constant term $a(0)$ of F at ∞ is non-zero.*

Remark 4.2.3.
1. In a previous version of [BS17], we proved Lemma 4.2.2 in a version which only worked for square free integers N, and hence had to include this restriction in all the results in this section. The above formulation for arbitrary N and the necessary adjustments in the proof are due to Jan Bruinier.
2. It is often possible to construct F as an eta quotient

$$\prod_{d|N} \eta(dz)^{r_d},$$

where $\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function. Such an eta quotient is a modular function for $\Gamma_0(N)$ if $\sum_{d|n} r_d = 0$, and if $\sum_{d|N} dr_d$ and $N \sum_{d|N} r_d/d$ are divisible by 24. Further, its order at a cusp $\frac{a}{c}$ is (independently of a) given by

$$\frac{1}{24} \sum_{d|N} \frac{(c,d)^2}{d} r_d, \tag{4.2.1}$$

which allows to search for suitable eta quotients using linear algebra. Unfortunately, it is not clear that the constant coefficient is non-zero, see the example in the next item. The author has checked that for all $N \leq 200$ there is a suitable eta quotient with integral Fourier coefficients at ∞, and we conjecture that one can always choose F as an eta quotient.

3. If $N = p$ is a prime, then $\Gamma_0(p)$ has only two cusps, represented by ∞ and 0. The function $\Delta(z)/\Delta(pz)$ has a pole at ∞ and vanishes at 0, and thus is a good candidate for a suitable F. Its constant coefficient is given by $\tau(p)$, where $\tau(n)$ are the coefficients of $\Delta(z)$. By Lehmer's conjecture, we expect that $\tau(p)$ never vanishes, but this is not known in general.

Proof of Lemma 4.2.2. If $N = 1$ and $k = 2$ then $M_k(\Gamma_0(N))$ is trivial. In this case we can take $F = 1$. Therefore we exclude this case from now on, so that $M_k(N)$ is non-trivial. We let $M_{k,0}(\Gamma_0(N)) \subset M_k(\Gamma_0(N))$ be the codimension 1 subspace of those modular forms which vanish at the cusp ∞.

Since the cusp at ∞ of $X_0(N)$ is defined over $\mathbb{Q}$, there exists an $E = \sum_{n\geq 0} c(n)q^n \in M_k(\Gamma_0(N))$ with rational coefficients which has value 1 at ∞, i.e., $c(0) = 1$. (Such an E can be obtained explicitly as a linear combination of the Eisenstein series at the cusps ∞ and 0.) It is well known that $M_k(\Gamma_0(N))$ has a basis consisting of modular forms with rational coefficients. Using E, we see that the space $M_{k,0}(\Gamma_0(N))$ also has a basis $g_1, \ldots, g_d$ consisting of forms with rational coefficients. Moreover, we have

$$M_k(\Gamma_0(N)) = M_{k,0}(\Gamma_0(N)) \oplus \mathbb{C}E.$$

The linear map $M_k(\Gamma_0(N)) \to \mathbb{C}[[q]]/\mathbb{C}[q]$ induced by mapping a modular form to its q-expansion is injective. Hence, the images of $g_1, \ldots, g_d$ and E are linearly independent. Consequently, there exists a polynomial

$$P_0 = \sum_{n<0} a(n)q^n \in q^{-1}\mathbb{Q}[q^{-1}]$$

such that

$$\{P_0, g_i\} = 0, \quad \text{for } i = 1, \ldots, d, \qquad \text{and} \qquad \{P_0, E\} = -1.$$

Put $P = P_0 + 1 \in \mathbb{Q}[q^{-1}]$. Then we have

$$\{P, g_i\} = 0, \quad \text{for } i = 1, \ldots, d, \qquad \text{and} \qquad \{P, E\} = 0,$$

and therefore $\{P, g\} = 0$ for all $g \in M_k(N)$.

According to Lemma 4.2.1 there exists an $F \in M^{!,\infty}_{2-k}(\Gamma_0(N))$ with principal part $P_F = P$. We denote the Fourier expansion of F by $F = \sum_n a(n)q^n$. The group $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on $M^!_{2-k}(\Gamma_0(N))$ by conjugation of the Fourier coefficients. Under the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on $X_0(N)$ the cusp at ∞ is fixed, and the other cusps are permuted among themselves. Hence $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ also acts on $M^{!,\infty}_{2-k}(\Gamma_0(N))$ by conjugation of the Fourier coefficients. Consequently, for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the form F^σ also belongs to $M^{!,\infty}_{2-k}(\Gamma_0(N))$. Since $P_F \in \mathbb{Q}[q^{-1}]$, the form $F - F^\sigma$ has vanishing principal part and therefore vanishes identically. We find that $a(n) = a(n)^\sigma$ for all $n \in \mathbb{Z}$. Therefore, all Fourier coefficients of F are rational. $\square$

We now use the modular function $F \in M^{!,\infty}_0(\Gamma_0(N))$ constructed in Lemma 4.2.2 as an input for the Kudla-Millson theta lift $I^{KM}(F,\tau)$ studied in [BF06] and the Millson theta lift $I^M(F,\tau)$ investigated above. The following theorem is just a straightforward simplification of Theorem 4.5 from [BF06] and Theorem 3.3.1 above. In order to simplify the formulas and the upcoming results, we multiply the expansion of the Millson lift given in Theorem 3.3.1 by $i/\sqrt{N}$.

Theorem 4.2.4. *Let $F(z) = \sum_{n \gg -\infty} a(n)q^n \in M^{!,\infty}_0(\Gamma_0(N))$ be as in Lemma 4.2.2 and $h \in \mathbb{Z}/2N\mathbb{Z}$.*

1. *The function*

$$I^{KM}(F,\tau)^+_h = \sum_{\substack{D\in\mathbb{Z}, D<0\\ D\equiv h^2(4N)}} \left(\mathrm{tr}^+_F(-D/4N,h) + \mathrm{tr}^-_F(-D/4N,h)\right)q^{-D/4N}$$
$$+ 4\delta_{0,h}\sum_{n\geq 0} a(-n)\sigma_1(n) - \sum_{b>0} b\left(\delta_{b,h} + \delta_{b,-h}\right)\sum_{n>0} a(-bn)q^{-b^2/4N}$$

is the h-th component of the holomorphic part of a harmonic Maass form of weight $3/2$ for ρ_L with

$$\xi_{3/2}\left(I^{KM}(F,\tau)\right) = -\frac{\sqrt{N}}{4\pi}a(0)\theta_{1/2}(\tau).$$

Here $\delta_{h,h'}$ equals 1 if $h \equiv h'(2N)$ and 0 otherwise, and $\sigma_1(0) = -\frac{1}{24}$.

2. *The function*

$$I^{M}(F,\tau)^+_h = \sum_{\substack{D\in\mathbb{Z}, D<0\\ D\equiv h^2(4N)}} \frac{i}{\sqrt{|D|}}\left(\mathrm{tr}^+_F(-D/4N,h) - \mathrm{tr}^-_F(-D/4N,h)\right)q^{-D/4N}$$
$$+ \sum_{b>0}\left(\delta_{b,h} - \delta_{b,-h}\right)\sum_{n>0} a(-bn)q^{-b^2/4N}$$

is the h-th component of the holomorphic part of a harmonic Maass form of weight $1/2$ for ρ_L with

$$\xi_{1/2}\left(I^{M}(F,\tau)\right) = -\frac{1}{2\sqrt{N}}a(0)\theta_{3/2}(\tau).$$

Remark 4.2.5. The Kudla-Millson lift of the constant 1-function gives a generalization of Zagier's non-holomorphic Eisenstein series of weight 3/2 from [Zag75] to arbitrary level N, see also [BF06], Remark 4.6. The ξ-image of the Eisenstein series is a linear combination of unary theta series associated to lattices $(\mathbb{Z}, n \mapsto -dn^2)$ with $d \mid N$, and is invariant under all Atkin-Lehner involutions. Since $\theta_{1/2}$ is only Atkin-Lehner invariant if $N = 1$ or $N = p$ is prime, we can **not** take the Eisenstein series as a ξ-preimage of $\theta_{1/2}$ in general. Also note that usually the principal parts of the harmonic Maass forms given above are non-zero.

By the theory of complex multiplication, the rationality properties of traces of weakly holomorphic modular functions are well understood. Therefore, we obtain the following result on the rationality of the holomorphic parts of the harmonic Maass forms given above.

Theorem 4.2.6. *Let $F \in M_0^!(\Gamma_0(N))$ and suppose that the Fourier coefficients of F at ∞ lie in $\mathbb{Z}$ and the expansions at all other cusps have coefficients in $\mathbb{Z}[\zeta_N]$. Then for $D \equiv h^2(4N), D < 0$, the numbers*

$$6\big(\operatorname{tr}_F^+(-D/4N, h) + \operatorname{tr}_F^-(-D/4N, h)\big) \tag{4.2.2}$$

and

$$6t\frac{i}{\sqrt{|D|}}\big(\operatorname{tr}_F^+(-D/4N, h) - \operatorname{tr}_F^-(-D/4N, h)\big) \tag{4.2.3}$$

are rational integers, where $D = t^2 D_0$ with a negative fundamental discriminant D_0.

Proof. The assumption on the integrality of F at ∞ implies that $F \in \mathbb{Q}(j, j_N)$. By the theory of complex multiplication (see Theorem 4.1 in [BO13]), the values $F(z_Q)$ of F at Heegner points z_Q of discriminant D lie in the ring class field of the order $\mathcal{O}_D$ over $\mathbb{Q}(\sqrt{D})$. Further, Lemma 4.3 in [BO13] asserts that the values $F(z_Q)$ are algebraic integers. The Galois group of the ring class field of $\mathcal{O}_D$ over $\mathbb{Q}(\sqrt{D})$ permutes the Heegner points occuring in $\operatorname{tr}_F^+(-D/4N, h)$ and $\operatorname{tr}_F^-(-D/4N, h)$, see [Gro84]. It follows that $6\operatorname{tr}_F^+(-D/4N, h)$ and $6\operatorname{tr}_F^-(-D/4N, h)$ are algebraic integers in $\mathbb{Q}(\sqrt{D})$, where the factor 6 was added to get rid of possible factors $|\overline{\Gamma}_0(N)_Q|$ in the denominator. Using that F has rational coefficients at ∞, we see that

$$\overline{\operatorname{tr}_F^-(-D/4N, h)} = \operatorname{tr}_F^+(-D/4N, h),$$

and thus

$$\operatorname{tr}_F^+(-D/4N, h) + \operatorname{tr}_F^-(-D/4N, h) \in \mathbb{Q}$$

and

$$\mathrm{tr}_F^+(-D/4N, h) - \mathrm{tr}_F^-(-D/4N, h) \in \sqrt{D}\mathbb{Q}.$$

This implies that the quantities in (4.2.2) and (4.2.3) are rational integers. $\square$

Remark 4.2.7. In all the numerical examples we looked at, the numbers in (4.2.2) and (4.2.3) were already integers without the factors 6 and $6t$. Possibly, this is always the case.

Combining Theorem 4.2.4 and Theorem 4.2.6 we obtain that if $F \in M_0^{!,\infty}(\Gamma_0(N))$ is as in Lemma 4.2.2 and has rational principal part at ∞, then the holomorphic parts of $I^{KM}(F,\tau)$ and $I^M(F,\tau)$ have rational Fourier coefficients. This rationality result is remarkable since the holomorphic coefficients of a harmonic Maass form of weight 1/2 which does not map to the space of unary theta functions under $\xi_{1/2}$ are conjectured to be transcendental almost always, see the conjecture and Corollary 1.4 in the introduction of [BO10b].

Theorem 4.2.8. *Let K be a number field and let $H_{k,\rho_L}(K)$ be the subspace of H_{k,ρ_L} consisting of forms whose principal part is defined over K.*

1. *Let $f \in H_{1/2,\rho_L}(K)$ and suppose that f is mapped to the space of unary theta functions by $\xi_{1/2}$. Then the coefficients of the holomorphic part f^+ of f lie in K.*
2. *Let $f \in H_{3/2,\rho_L}(K)$ and suppose that f is mapped to the space of unary theta functions by $\xi_{3/2}$. Then there is a cusp form $f' \in S_{3/2,\rho_L}$ such that the coefficients of $f^+ - f'$ lie in K.*

Remark 4.2.9. The corresponding statement for the spaces $H_{1/2,\rho_L^*}$ and $H_{3/2,\rho_L^*}$ is also true, but a little less interesting. Since there are no unary theta functions for ρ_L, it just says that a weakly holomorphic modular form of weight 1/2 or 3/2 for ρ_L^*, whose principal part is defined over K, has coefficients in K up to addition of a cusp form. This follows immediatly from the fact that the spaces $M^!_{1/2,\rho_L^*}$ and $M^!_{3/2,\rho_L^*}$ have bases with rational coefficients (see [McG03]).

Proof of Theorem 4.2.8. We only prove the first claim, since the second one is similar. Let $f \in H_{1/2,\rho_L}(K)$ and suppose that $\xi_{1/2}f$ lies in the space of unary theta functions. By Theorem 4.2.4, there is some $h \in H_{1/2,\rho_L}$, which is a linear combination of harmonic Maass forms h_j with rational holomorphic parts, such that $\xi_{1/2}f = \xi_{1/2}h$, i.e., $f - h$ is weakly holomorphic. We can write $f-h$ as a linear combination of forms g_i with rational coefficients (see [McG03]). Having f written in terms of the g_i and h_j, we consider the system of linear equations obtained from comparing the principal parts. It is defined over K and has a solution in $\mathbb{C}$, so we can also solve it over K. Thereby we obtain a harmonic Maass form $\tilde{f}$ that has the same principal part as f and still maps to the space of unary theta functions under $\xi_{1/2}$, but is now a linear combination of the g_i and h_j over K. In particular, the coefficients of $\tilde{f}^+$ lie in K. Then $f - \tilde{f}$ is a harmonic Maass forms which has vanishing principal part and maps to the space of unary theta function under $\xi_{1/2}$. By Lemma 2.3.6, this implies that $f - \tilde{f}$ is a cusp form. But $S_{1/2,\rho_L} \cong J_{1,N}^{\mathrm{cusp}} = \{0\}$, so $f = \tilde{f}$, and thus f^+ has coefficients in K. $\square$

4.3 Regularized inner products and Weyl vectors of Borcherds products

The harmonic Maass forms constructed in Theorem 4.2.4 can be used to evaluate the regularized Petersson inner product of the unary theta functions $\theta_{1/2}$ and $\theta_{3/2}$ with harmonic Maass forms whose shadows are cusp forms.

Theorem 4.3.1. *Let $F(z) = \sum_{n \gg -\infty} a(n) q^n \in M_0^{!,\infty}(\Gamma_0(N))$ be as in Lemma 4.2.2.*

1. *Let $f \in H_{1/2,\rho_L^*}$ with holomorphic coefficients $c_f^+(D/4N, h)$, where $D \equiv h^2 (4N)$, and suppose that $\xi_{1/2} f \in S_{3/2,\rho_L}$. Then*

$$\begin{aligned}
&-\frac{\sqrt{N}}{4\pi} a(0) (f, \theta_{1/2})^{\mathrm{reg}} \\
&= \sum_{h(2N)} \sum_{\substack{D \in \mathbb{Z}, D<0 \\ D \equiv h^2(4N)}} c_f^+(D/4N, h) \left(\mathrm{tr}_F^+(-D/4N, h) + \mathrm{tr}_F^-(-D/4N, h) \right) \\
&\quad + 4 c_f^+(0,0) \sum_{n \geq 0} a(-n) \sigma_1(n) - 2 \sum_{b>0} c_f^+(b^2/4N, b) b \sum_{n>0} a(-bn).
\end{aligned}$$

2. *Let $f \in H_{3/2,\rho_L^*}$ with holomorphic coefficients $c_f^+(D/4N, h)$, where $D \equiv h^2 (4N)$, and suppose that $\xi_{3/2} f \in S_{1/2,\rho_L}$. Then*

$$\begin{aligned}
&-\frac{1}{2\sqrt{N}} a(0) (f, \theta_{3/2})^{\mathrm{reg}} \\
&= \sum_{h(2N)} \sum_{\substack{D \in \mathbb{Z}, D<0 \\ D \equiv h^2(4N)}} c_f^+(D/4N, h) \frac{i}{\sqrt{|D|}} \left(\mathrm{tr}_F^+(-D/4N, h) - \mathrm{tr}_F^-(-D/4N, h) \right) \\
&\quad + 2 \sum_{b>0} c_f^+(b^2/4N, b) \sum_{n>0} a(-bn).
\end{aligned}$$

Proof. We show the formula for $\theta_{1/2}$. Using Stokes' theorem, we see as in the proof of

[BF04], Proposition 3.5, that

$$(f,\theta_{1/2})^{\mathrm{reg}} = -(I^{KM}(F,\tau),\xi_{1/2}f)^{\mathrm{reg}} + \lim_{T\to\infty}\int_{-1/2}^{1/2}\langle f(u+iT), \overline{I^{KM}(F,u+iT)}\rangle du.$$

One can show as in [Alf14], Theorem 5.1, that $I^{KM}(F,\tau)$ and $I^{M}(F,\tau)$ are orthogonal to cusp forms, i.e., $(I^{KM}(F,\tau),\xi_{1/2}f)^{\mathrm{reg}} = 0$. The integral on the right-hand side picks out the zero-coefficient, to which only the holomorphic part of f contributes in the limit. Hence we obtain a formula for $(f,\theta_{1/2})^{\mathrm{reg}}$ of the shape (2.3.7), involving only the coefficients of f^+. Plugging in the coefficients of $I^{KM}(F,z)$ from Theorem 4.2.4 yields the result. □

Example 4.3.2. As a simple application of the last result, we show that the Petersson norms of $\theta_{1/2}$ and $\theta_{3/2}$ are given by

$$(\theta_{1/2},\theta_{1/2}) = \frac{\pi(N+1)}{3\sqrt{N}} \qquad \text{and} \qquad (\theta_{3/2},\theta_{3/2}) = \frac{\sqrt{N}(N-1)}{6}.$$

These can of course also be evaluated using more direct methods, for instance, the Rankin-Selberg L-function, but it is interesting to see how the dependency on F in Theorem 4.3.1 disappears if we plug in $\theta_{1/2}$ or $\theta_{3/2}$ for f.

We only show the formula for $\theta_{1/2}$, since the proof for $\theta_{3/2}$ is very similar. We can assume $a(0) = 1$. Let

$$E_2^*(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)e(nz) - \frac{\pi}{3y}, \qquad \left(\sigma_1(n) = \sum_{d\mid N} d\right),$$

be the non-holomorphic Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. Then $E_2^*(z)-NE_2^*(Nz)$ is a holomorphic modular form of weight 2 for $\Gamma_0(N)$, and by applying the residue theorem to $F(z)(E_2^*(z)-NE_2^*(Nz))dz$, we find that F satisfies

$$(1-N) - 24\sum_{n>0} a(-n)(\sigma_1(n) - N\sigma_1(n/N)) = 0. \tag{4.3.1}$$

If we denote by $c_\theta(D/4N,h)$ the coefficients of $\theta_{1/2}$, we see that $c_\theta(0,0) = 1$, and $c_\theta(b^2/4N,b)$ equals 2 or 1 for $b > 0$ depending on whether $b \equiv -b(2N)$ or not. Applying Theorem 4.3.1 and using the relation (4.3.1) we obtain

$$-\frac{\sqrt{N}}{4\pi}(\theta_{1/2},\theta_{1/2}) = 4\sum_{n\geq 0} a(-n)\sigma_1(n) - 4\sum_{\substack{b>0\\ b\equiv 0(N)}} b\sum_{n>0} a(-bn) - 2\sum_{\substack{b>0\\ b\not\equiv 0(N)}} b\sum_{n>0} a(-bn)$$
$$= 4\sigma_1(0) + 2\sum_{n>0} a(-n)(\sigma_1(n) - N\sigma_1(n/N)) = -\frac{1+N}{12}.$$

This yields the stated formula.

The formula given in the first item of Theorem 4.3.1 has applications in the theory of Borcherds products, see [Bor98]. We follow the exposition of [BO10b]. Let $f \in H_{1/2,\rho_L^*}$ be a harmonic Maass form of weight 1/2 for ρ_L^* whose shadow is a cusp form, and assume that $c_f^+(D/4N,h) \in \mathbb{R}$ for all D and $c_f^+(D/4N,h) \in \mathbb{Z}$ for $D \leq 0$. Then the infinite product

$$\Psi(z,f) = e(\rho_{f,\infty}z)\prod_{n=1}^{\infty}(1-e(nz))^{c_f^+(n^2/4N,n)}$$

is a meromorphic modular form of weight $c_f^+(0,0)$ for $\Gamma_0(N)$ and a unitary character, possibly of infinite order (see Theorems 6.1 and 6.2 in [BO10b]). Here $\rho_{f,\infty}$ is the so-called **Weyl vector** at ∞, which is defined by

$$\rho_{f,\infty} = \frac{\sqrt{N}}{8\pi}(f,\theta_{1/2})^{\mathrm{reg}}.$$

The Bocherds product $\Psi(z,f)$ has singularities at Heegner points in $\mathbb{H}$, which are prescribed by the principal part of f, and its orders at the cusps are determined by the corresponding Weyl vectors, which we describe now.

Each cusp of $\Gamma_0(N)$ can be represented by a reduced fraction a/c with $c \mid N$, and the Weyl vector corresponding to a/c is defined by

$$\rho_{f,a/c} = \frac{\sqrt{N}}{8\pi}(f,\theta_{1/2,N/(c,N/c)^2}^{\sigma_{c/(c,N/c)}}|U_{(c,N/c)})^{\mathrm{reg}}, \tag{4.3.2}$$

where $\sigma_{c/(c,N/c)}$ denotes the Atkin-Lehner involution corresponding to the exact divisor $c/(c,N/c)$ of $N/(c,N/c)^2$ as in (2.3.9), and $U_{(c,N/c)}$ is the level raising operator (2.3.10). Note that the Weyl vector at a/c does not depend on a. Further, Theorem 4.3.1 yields a formula for the Weyl vector at each cusp a/c, involving only the principal part of f and the coefficients $c_f^+(b^2,r)$ for $b>0$ and $r \in \mathbb{Z}/2N\mathbb{Z}$ with $r^2 \equiv b^2(4N)$. Thus, we obtain the following rationality result.

Corollary 4.3.3. *Let $f \in H_{1/2,\rho_L^*}$ be a harmonic Maass form with $\xi_{1/2}f \in S_{3/2,\rho_L}$. Suppose that $c_f^+(D/4N,h) \in \mathbb{R}$ for all D and that $c_f^+(D/4N,h) \in \mathbb{Z}$ for $D \leq 0$. If*

$c_f^+(b^2/4N,h) \in \mathbb{Q}$ for all $b>0$ and all possible $h \in \mathbb{Z}/2N\mathbb{Z}$, then the Weyl vectors $\rho_{f,a/c}$ at all cusps are rational.

The formula for the Weyl vector $\rho_{f,a/c}$ obtained from Theorem 4.3.1 looks quite complicated in general. Thus, for simplicity, we only state it in the special case of a cusp a/c with $c \mid\mid N$ and $(a,c)=1$. Then $\theta_{1/2,N/(c,N/c)^2}^{\sigma_{c/(c,N/c)}}|U_{(c,N/c)} = \theta_{1/2,N}^{\sigma_c}$, and Theorem 4.3.1 gives the following formula.

Corollary 4.3.4. *Let $f \in H_{1/2,\rho_L^*}$ be a harmonic Maass form with $\xi_{1/2}f \in S_{3/2,\rho_L}$. Suppose that $c_f^+(D/4N,h) \in \mathbb{R}$ for all D and that $c_f^+(D/4N,h) \in \mathbb{Z}$ for $D \leq 0$. Let $c \mid\mid N$ and let σ_c be the associated Atkin-Lehner involution as in (2.3.9). Let $F \in M_0^{!,\infty}(N)$ be as in Lemma 4.2.2, normalized to $a(0)=1$. Then the Weyl vector $\rho_{f,a/c}$ at the cusp a/c is given by*

$$\begin{aligned}
\rho_{f,a/c} &= \frac{\sqrt{N}}{8\pi}(f^{\sigma_c},\theta_{1/2})^{\mathrm{reg}} \\
&= -\frac{1}{2}\sum_{h(2N)}\sum_{\substack{D<0\\ D\equiv h^2(4N)}} c_f^+(D/4N,\sigma_c(h))\left(\mathrm{tr}_F^+(-D/4N,h)+\mathrm{tr}_F^-(-D/4N,h)\right) \\
&\quad - 2c_f^+(0,0)\sum_{n\geq 0} a(-n)\sigma_1(n) + \sum_{b>0} c_f^+(b^2/4N,\sigma_c(b))b\sum_{n>0} a(-bn).
\end{aligned}$$

Remark 4.3.5.

1. If N is square free, the cusps of $\Gamma_0(N)$ are represented by the fractions $1/c$, where c runs through the divisors of N. In this case all Weyl vectors can be computed with the above formula.

2. In [Bor98], Section 9, the Weyl vectors are computed in a similar way, using non-holomorphic Eisenstein series of weight $3/2$ as ξ-preimages for $\theta_{1/2}$. However, this only works if $N=1$ or if $N=p$ is a prime. Otherwise, the Eisenstein series, and thus also its ξ-image, is invariant under all Atkin-Lehner involutions, but $\theta_{1/2}$ is not.

Example 4.3.6. We consider the Borcherds lift of $f=\theta_{1/2}$, for N arbitrary. By Example 4.3.2, the Weyl vector of $\theta_{1/2}$ at ∞ equals $(1+N)/24$, so its Borcherds product is given by

$$\Psi(z,\theta_{1/2}) = \eta(z)\eta(Nz).$$

By a similar computation as in Example 4.3.2 we find

$$(\theta_{1/2}^{\sigma_c},\theta_{1/2}) = \frac{\pi}{3\sqrt{N}}\left(\frac{N}{c}+c\right)$$

for $c \mid\mid N$. Hence the Weyl vector of $\Psi(z,f)$ at a cusp a/c with $c \mid\mid N$ is given by

$\frac{1}{24}\frac{N}{c}\left(1+\frac{c^2}{N}\right)$. From this we can infer that the Borcherds product of $\theta_{1/2}^{\sigma_c}$ equals

$$\Psi(z, \theta_{1/2}^{\sigma_c}) = \eta(cz)\eta\left(\frac{N}{c}z\right).$$

Finally, we would like to mention that the harmonic Maass form $I^M(F,\tau)$ given in Theorem 4.2.4 can be used to construct rational functions on $X_0(N)$ with special divisors. Let $\Delta \neq 1$ be a fundamental discriminant and let $r \in \mathbb{Z}$ with $\Delta \equiv h^2(4N)$. Further, let $\tilde{\rho}_L = \rho_L$ if $\Delta > 1$ and $\tilde{\rho}_L = \rho_L^*$ if $\Delta < 0$ for the moment. The twisted Borcherds product of a harmonic Maass form $f \in H_{1/2,\tilde{\rho}_L^*}$ with real holomorphic part and integral principal part is defined by

$$\Psi_{\Delta,r}(z,f) = \prod_{n=1}^{\infty}\prod_{b(\Delta)}[1 - e(b/\Delta)e(nz)]^{\left(\frac{\Delta}{b}\right)c_f^+(|\Delta|n^2/4N, rn)},$$

see [BO10b], Theorem 6.1. Note that the Weyl vectors vanish for $\Delta \neq 1$. The function $\Psi_{\Delta,r}(z,f)$ is a meromorphic modular form of weight 0 for $\Gamma_0(N)$ and a unitary character, which is of finite order if and only if the coefficients $c_f^+(|\Delta|n^2/4N, rn)$ are rational (see Theorem 6.2 in [BO10b]).

If $F \in M_0^{!,\infty}(N)$ is as in Lemma 4.2.2 and has integral principal part, then the Millson lift $I^M(F,\tau)$ given in Theorem 4.2.4 is a harmonic Maass form in $H_{1/2,\rho_L}$ with rational holomorphic part and integral principal part. In particular, for $\Delta < 0$, some power of the twisted Borcherds product $\Psi_{\Delta,r}(z, I^M(F,\tau))$ defines a rational function on $X_0(N)$ whose zeros and poles lie on a twisted Heegner divisor.

5 Borchers Lifts of Harmonic Maass Forms

In this chapter, we extend Borcherds' regularized theta lift from weight $1/2$ weakly holomorphic modular forms to real analytic modular functions with logarithmic singularities to twisted Borcherds lifts of harmonic Maass forms of weight $1/2$.

5.1 Analytic properties of the Borcherds lift

Let Δ be a fundamental discriminant (possibly 1) and let $r \in \mathbb{Z}/2N\mathbb{Z}$ such that $\Delta \equiv r^2 \bmod 4N$. Recall from Section 2.4.4 the notations

$$\tilde{\rho}_L = \begin{cases} \rho_L, & \text{if } \Delta > 0, \\ \rho_L^*, & \text{if } \Delta < 0, \end{cases} \qquad Q_\Delta(X) = \frac{1}{|\Delta|}Q(X), \qquad (X,Y)_\Delta = \frac{1}{|\Delta|}(X,Y),$$

and the twisted Siegel theta function

$$\Theta_{\Delta,r}(\tau,z) = v \sum_{h \in L'/L} \sum_{\substack{X \in L+rh \\ Q(X) \equiv \Delta Q(h)\,(\Delta)}} \chi_\Delta(X) e\big(\tau Q_\Delta(X_z) + \bar{\tau} Q_\Delta(X_{z^\perp})\big)\mathfrak{e}_h, \tag{5.1.1}$$

which is $\Gamma_0(N)$-invariant in z and transforms like a modular form of weight $-1/2$ for $\tilde{\rho}_L$. Let $f \in H_{1/2,\tilde{\rho}_L^*}$. In this chapter, it is more convenient to work with the normalization of the Fourier expansion of f given in (2.3.2), involving the functions $\beta_{1/2}(w)$ and $\beta_{1/2}^c(w)$.

We let

$$H_{\Delta,r}^+(f) = \bigcup_{\substack{h \in L'/L, n<0 \\ c_f^+(n,h) \neq 0}} \left\{ z_X : X \in L_{-|\Delta|n,rh} \right\}, \qquad H_{\Delta,r}^-(f) = \bigcup_{\substack{h \in L'/L, n>0 \\ c_f^-(n,h) \neq 0}} \bigcup_{X \in L_{-|\Delta|n,rh}} c_X,$$

be the sets of Heegner points and geodesics associated to f.

Following Borcherds [Bor98], we define the regularized theta lift of $f \in H_{1/2,\tilde{\rho}_L^*}$ by

$$\Phi_{\Delta,r}(f,z) = \mathrm{CT}_{s=0}\left(\lim_{T \to \infty} \int_{\mathcal{F}_T} \left\langle f(\tau), \overline{\Theta_{\Delta,r}(\tau,z)} \right\rangle v^{-s} \frac{du\,dv}{v^2} \right), \tag{5.1.2}$$

where

$$\mathcal{F}_T = \{\tau = u + iv \in \mathbb{H} : |\tau| \geq 1, |u| \leq 1/2, v \leq T\}$$

is a truncated fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and $\mathrm{CT}_{s=0} F(s)$ denotes the constant term in the Laurent expansion of the analytic continuation of $F(s)$ at $s = 0$. Borcherds [Bor98] proved that for $\Delta = 1$ and a weakly holomorphic modular form $f \in M^!_{1/2,\rho_L^*}$ the above regularized integral exists and defines a real analytic $\Gamma_0(N)$-invariant function with logarithmic singularities at the Heegner points in $H^+_{\Delta,r}(f)$. It was shown by Bruinier and Ono [BO10b] that this result remains true for the twisted lifts of harmonic Maass forms $f \in H^+_{1/2,\tilde{\rho}_L^*}$ which map to cusp forms under $\xi_{1/2}$. We will generalize the theta lift to the full space $H_{1/2,\tilde{\rho}_L^*}$, and we will see that further singularities along the geodesics in $H^-_{\Delta,r}(f)$ occur.

We say that a complex-valued function f defined on some subset of $\mathbb{R}^n$ has a singularity of type g (written $f \approx g$) at a point z_0 if there is an open neighbourhood U of z_0 such that f and g are defined on a dense subset of U and $f - g$ can be continued to a real analytic function on U.

Theorem 5.1.1. *For $f \in H_{1/2,\tilde{\rho}_L^*}$ the Borcherds lift $\Phi_{\Delta,r}(f,z)$ defines a $\Gamma_0(N)$-invariant real analytic function on $\mathbb{H} \setminus (H^+_{\Delta,r}(f) \cup H^-_{\Delta,r}(f))$ with*

$$\Delta_0 \Phi_{\Delta,r}(f,z) = \begin{cases} -2a_f^+(0,0), & \text{if } \Delta = 1, \\ 0, & \text{if } \Delta \neq 1. \end{cases}$$

At a point $z_0 \in H^+_{\Delta,r}(f) \cup H^-_{\Delta,r}(f)$ it has a singularity of type

$$- \sum_{h \in L'/L} \sum_{n<0} c_f^+(n,h) \sum_{\substack{X \in L_{-|\Delta|n,rh} \\ z_0 = z_X}} \chi_\Delta(X) \log(-Q_\Delta(X_{z^\perp}))$$

$$+ \sum_{h \in L'/L} \sum_{n>0} c_f^-(n,h) n^{-1/2} \sum_{\substack{X \in L_{-|\Delta|n,rh} \\ z_0 \in c_X}} \chi_\Delta(X) \arcsin\left(\sqrt{\frac{Q_\Delta(X)}{Q_\Delta(X_{z^\perp})}}\right).$$

Remark 5.1.2.

1. Recall that for $X = \begin{pmatrix} -b/2N & -c/N \\ a & b/2N \end{pmatrix} \in L'$ we have

$$-Q_\Delta(X_{z^\perp}) = \frac{1}{4N|\Delta|y^2}|Q_X(z)|^2 = \frac{1}{4N|\Delta|y^2}|aNz^2 + bz + c|^2,$$

which yields a more explicit formula for the singularities. Since $0 < \frac{Q_\Delta(X)}{Q_\Delta(X_{z^\perp})} \leq 1$ for all X with $Q_\Delta(X) < 0$, and $Q_\Delta(X)/Q_\Delta(X_{z^\perp}) = 1$ exactly for $z \in c_X$, we see that the Borcherds lift extends to a continuous function on $\mathbb{H} \setminus H^+_{\Delta,r}(f)$, which is not differentiable along the geodesics in $H^-_{\Delta,r}(f)$. Note that we can also write the singularities in the form

$$\arcsin\left(\sqrt{\frac{Q_\Delta(X)}{Q_\Delta(X_{z^\perp})}}\right) = \arctan\left(\sqrt{\frac{Q_\Delta(X)}{-Q_\Delta(X_z)}}\right).$$

2. For $d \,||\, N$ the Atkin-Lehner involution W_d acts on the Siegel theta function by

$$\Theta_{\Delta,r}(\tau, W_d z) = \Theta_{\Delta,r}(\tau, z)^{w_d},$$

which implies that the Borcherds lift satisfies

$$\Phi_{\Delta,r}(f, W_d z) = \Phi_{\Delta,r}(f^{w_d}, z)$$

Proof of Theorem 5.1.1. We first show that for $z \in \mathbb{H} \setminus (H^+_{\Delta,r}(f) \cup H^-_{\Delta,r}(f))$ the integral in (5.1.2) converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1/2$ and has a meromorphic continuation to $s = 0$. The proof follows the arguments of [Bru02], Proposition 2.8.

The integral over the compact set $\mathcal{F}_1 = \{\tau \in \mathbb{H} : |\tau| \geq 1, |u| \leq 1/2, v \leq 1\}$ converges absolutely and locally uniformly for all $s \in \mathbb{C}$ and $z \in \mathbb{H}$. We consider the remaining integral

$$\varphi(z, s) = \int_{v=1}^{\infty}\int_{u=0}^{1} \left\langle f(\tau), \overline{\Theta_{\Delta,r}(\tau, z)}\right\rangle v^{-s}\frac{du\,dv}{v^2}.$$

Inserting the Fourier expansions of $f(\tau)$ and $\Theta_{\Delta,r}(\tau, z)$ and carrying out the integral over u, we obtain

$$\begin{aligned}
\varphi(z, s) &= \chi_\Delta(0)\left(c^+_f(0,0)\int_{v=1}^{\infty} v^{-1-s}dv + c^-_f(0,0)\int_{v=1}^{\infty} v^{-1/2-s}dv\right)\\
&+ \int_{v=1}^{\infty}\sum_{h,X}\chi_\Delta(X)c^+_f(-Q_\Delta(X), h)\exp\left(4\pi Q_\Delta(X_{z^\perp})v\right)v^{-s-1}dv\\
&+ \int_{v=1}^{\infty}\sum_{Q(X)=0}\chi_\Delta(X)c^-_f(-Q_\Delta(X), h)\exp(4\pi Q_\Delta(X_{z^\perp})v)v^{-s-1/2}dv\\
&+ \int_{v=1}^{\infty}\sum_{Q(X)>0}\chi_\Delta(X)c^-_f(-Q_\Delta(X), h)\beta_{1/2}(4\pi Q_\Delta(X)v)\exp(4\pi Q_\Delta(X_{z^\perp})v)v^{-s-1/2}dv\\
&+ \int_{v=1}^{\infty}\sum_{Q(X)<0}\chi_\Delta(X)c^-_f(-Q_\Delta(X), h)\beta^c_{1/2}(4\pi Q_\Delta(X)v)\exp(4\pi Q_\Delta(X_{z^\perp})v)v^{-s-1/2}dv
\end{aligned}$$

where the sums run over $h \in L'/L$ and $X \in (L + rh) \setminus \{0\}$ with $Q(X) \equiv \Delta Q(h) \bmod \Delta$.

Since $\chi_\Delta(0) = 0$ for $\Delta \neq 1$ the integrals in the first line only appear if $\Delta = 1$. They can be evaluated for $\mathrm{Re}(s) > 1/2$ by

$$\int_{v=1}^{\infty} v^{-1-s} dv = \frac{1}{s}, \qquad \int_{v=1}^{\infty} v^{-1/2-s} dv = \frac{1}{s-1/2},$$

giving their meromorphic continuations to $s = 0$. Note that this shows that for $\Delta = 1$ the regularization involving the extra parameter s is really necessary.

The integral in the second line involving the coefficients $c_f^+(n,h)$ converges locally uniformly and absolutely for $s \in \mathbb{C}$ and $z \in \mathbb{H} \setminus H_{\Delta,r}^+(f)$ by the same arguments as in the proof of [Bru02], Proposition 2.8. The integrals over the sums corresponding to $Q(X) = 0$ and $Q(X) > 0$ in the third and fourth line can be treated in the same way, and they converge locally uniformly and absolutely for $s \in \mathbb{C}$ and $z \in \mathbb{H}$.

The remaining integral in the fifth line can be written as

$$\sum_{h \in L'/L} \sum_{n>0} c_f^-(n,h) \int_{v=1}^{\infty} \sum_{X \in L_{-|\Delta|n,rh}} \chi_\Delta(X) \beta_{1/2}^c(4\pi Q_\Delta(X) v) \exp\left(4\pi Q_\Delta(X_{z^\perp}) v\right) v^{-s-1/2} dv,$$

where the first two sums are finite. Hence, estimating

$$\beta_{1/2}^c(4\pi Q_\Delta(X)v) \leq 2\exp(-4\pi Q_\Delta(X)v)$$

and using $Q_\Delta(X) = Q_\Delta(X_z) + Q_\Delta(X_{z^\perp})$, it suffices to consider the integral

$$\int_{v=1}^{\infty} \sum_{X \in L_{-|\Delta|n,rh}} \exp\left(-4\pi Q_\Delta(X_z) v\right) v^{-\mathrm{Re}(s)-1/2} dv. \tag{5.1.3}$$

For any $C \geq 0$ and any compact subset $K \subset \mathbb{H}$ the set

$$\left\{ X \in L_{-|\Delta|n,rh} : \exists z \in K \text{ with } |Q_\Delta(X_z)| \leq C \right\}$$

is finite, so if $z \in K \subset \mathbb{H} \setminus H_{\Delta,r}^-(f)$ then there is some $\varepsilon > 0$ such that $Q_\Delta(X_z) > \varepsilon$ for all $X \in L_{-|\Delta|n,rh}$. We can now estimate

$$\sum_{X \in L_{-|\Delta|n,rh}} \exp\left(-4\pi Q_\Delta(X_z)v\right) \leq e^{-2\pi\varepsilon v} e^{\pi n} \sum_{X \in L_{-|\Delta|n,rh}} \exp\left(-\pi(Q_\Delta(X_z) - Q_\Delta(X_{z^\perp}))\right)$$

for $v \geq 1$. The series on the right-hand side converges since $X \mapsto Q_\Delta(X_z) - Q_\Delta(X_{z^\perp})$ is a positive definite quadratic form. In particular, the integral in (5.1.3) converges absolutely and locally uniformly for $s \in \mathbb{C}$ and $z \in \mathbb{H} \setminus H_{\Delta,r}^-(f)$. This shows that the regularized theta integral exists.

By similar arguments as above we see that all iterated partial derivatives of $\Phi_{\Delta,r}(f,z)$ converge absolutely and locally uniformly on $\mathbb{H} \setminus (H_{\Delta,r}^+(f) \cup H_{\Delta,r}^-(f))$, so the Borcherds lift is a smooth function. The statement concerning the Laplacian can now be proven by interchanging $\Delta_0 = \Delta_{0,z}$ with the integral, using the differential equation

$$\Delta_{0,z}\Theta_{\Delta,r}(\tau,z) = 4v^{1/2}\overline{\Delta_{1/2,\tau}v^{-1/2}\overline{\Theta_{\Delta,r}(\tau,z)}},$$

(see Lemma 2.4.8) and then applying Stokes' theorem to move $\Delta_{1/2,\tau}$ from the theta function to $f(\tau)v^{-s}$ in the integral (compare [Bru02], Lemma 4.3). It is easy to verify that the appearing boundary integrals vanish. By computing $\Delta_{1/2}(f(\tau)v^{-s})$ explicitly and using that f is harmonic, we obtain

$$\Delta_0\Phi_{\Delta,r}(f,z) = -2\operatorname{Res}_{s=0}\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle f(\tau),\overline{\Theta_{\Delta,r}(\tau,z)}\rangle v^{-s}\frac{du\,dv}{v^2}.$$

We have seen above that the integral on the right-hand side is holomorphic at $s=0$ if $\Delta\neq 1$, and has a simple pole with residue $a_f^+(0,0)$ if $\Delta = 1$, coming from the first integral in the first line of $\varphi(z,s)$. This shows the Laplace equation for $\Phi_{\Delta,r}(f,z)$, which also implies that the Borcherds lift is real analytic by a standard regularity result for elliptic differential equations.

The singularities of $\Phi_{\Delta,r}(f,z)$ can be determined using the following lemma with $n = -Q_\Delta(X)$ and $t = -Q_\Delta(X_{z^\perp})$. □

Lemma 5.1.3. *1. The function*

$$I^+(t) = \int_{v=1}^\infty e^{-4\pi tv}\frac{dv}{v}$$

is real analytic for $t>0$ and has a singularity of type $-\log(t)$ at $t=0$.

2. For $n>0$ the function

$$I_n^-(t) = \int_{v=1}^\infty \sqrt{v}\beta_{1/2}^c(-4\pi nv)e^{-4\pi tv}\frac{dv}{v}$$

is real analytic for $t>n$ and has a singularity of type $n^{-1/2}\arcsin\left(\sqrt{\frac{n}{t}}\right)$ at $t=n$.

Proof. We follow the proof of [Bor98], Lemma 6.1. Using partial integration and the fact that $\log(v)$ is integrable near $v=0$, we see that

$$I^+(t) \approx 4\pi t\int_{v=0}^\infty e^{-4\pi tv}\log(v)dv = \int_{v=0}^\infty e^{-v}\log\left(\frac{v}{4\pi t}\right)dv \approx -\log(t).$$

For $n > 0$, we use that $\sqrt{v}\beta^c_{1/2}(-4\pi nv) = O(\sqrt{v})$ as $v \to 0$ and compute

$$\begin{aligned} I_n^-(t) &\approx \int_{v=0}^{\infty} \left(2\sqrt{v}\int_{w=0}^{1} e^{4\pi nvw^2}dw\right) e^{-4\pi tv}\frac{dv}{v} \\ &= \int_{w=0}^{1} \frac{1}{\sqrt{t-nw^2}}dw \\ &= n^{-1/2}\arcsin\left(\sqrt{\frac{n}{t}}\right). \end{aligned}$$

This finishes the proof of the lemma and of Theorem 5.1.1. □

5.2 The Fourier expansion of the Borcherds lift

Next, we compute the Fourier expansion of the Borcherds lift. To this end, we first need to introduce a special function which captures the arcsin singularities of $\Phi_{\Delta,r}(f,z)$ along vertical geodesics.

For $a \geq 1$ and $\mathrm{Re}(s) > -1$ we define

$$\arcsin_s\left(\frac{1}{\sqrt{a}}\right) = \int_0^1 \frac{1}{\sqrt{a-t^2}}\left(\frac{1-t^2}{a-t^2}\right)^s dt. \tag{5.2.1}$$

The function $\arcsin_s$ is holomophic in s and satisfies

$$\arcsin_0(1/\sqrt{a}) = \arcsin(1/\sqrt{a}).$$

The factor $(1-t^2)^s$ ensures that the integral converges at $a = 1$ if $\mathrm{Re}(s) \geq 1/2$, and the factor $(a-t^2)^s$ in the denominator was added to make the estimate

$$|\arcsin_s(1/\sqrt{a})| \leq (a-1)^{-\mathrm{Re}(s)-1/2} \tag{5.2.2}$$

for $a > 1$ and $\mathrm{Re}(s) > 0$ hold. Note that for $\mathrm{Re}(s) > -1$ we can write

$$\arcsin_s\left(\frac{1}{\sqrt{a}}\right) = \frac{\sqrt{\pi}\,\Gamma(s+1)}{2\Gamma(s+1/2)}B(1/a; s+1/2, 1/2), \tag{5.2.3}$$

where

$$B(z;\alpha,\beta) = \int_0^z u^{\alpha-1}(1-u)^{\beta-1}du$$

is the incomplete beta function. This representation makes it easier to compute the derivative of $\arcsin_s$.

Lemma 5.2.1. *For $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > 0$ we have the Fourier expansion*

$$\sum_{\ell \in \mathbb{Z}} \operatorname{arcsin}_s\left(\frac{y}{\sqrt{(x+\ell)^2+y^2}}\right) = y\frac{\sqrt{\pi}\,\Gamma(s)}{\Gamma(s+1/2)}$$
$$+ 2y\frac{\sqrt{\pi}}{\Gamma(s+1/2)}\sum_{n\neq 0}(\pi|n|y)^s\left(\int_0^1 (1-t^2)^{s/2}K_s\left(2\pi|n|y\sqrt{1-t^2}\right)dt\right)\cos(2\pi n x),$$

where K_s denotes the K-Bessel function of order s. For $\mathrm{Re}(s) > -1$ the series on the right-hand side converges absolutely and locally uniformly in s. In particular, the left-hand side has a meromorphic continuation to $\mathrm{Re}(s) > -1$ with a simple pole at $s = 0$.

Proof. The estimate (5.2.2) shows that the series on the left-hand side converges absolutely for $\mathrm{Re}(s) > 0$. It is 1-periodic and even in x and hence has a Fourier expansion of the form $\sum_{n\in\mathbb{Z}} a(n,y)\cos(2\pi n x)$ with coefficients

$$a(n,y) = \int_{-\infty}^{\infty} \operatorname{arcsin}_s\left(\frac{1}{\sqrt{(u/y)^2+1}}\right)\cos(2\pi n u)du.$$

We plug in the definition of arcsin_s and interchange the order of integration to find

$$\begin{aligned} a(n,y) &= \int_0^1\left(\int_{-\infty}^{\infty}\frac{\cos(2\pi n u)}{((u/y)^2+1-t^2)^{s+1/2}}du\right)(1-t^2)^s dt \\ &= y\int_0^1\left(\int_{-\infty}^{\infty}\frac{\cos\left(2\pi n u y\sqrt{1-t^2}\right)}{(u^2+1)^{s+1/2}}du\right)dt. \end{aligned}$$

For $n = 0$ the inner integral can be evaluated as

$$\int_{-\infty}^{\infty}\frac{1}{(u^2+1)^{s+1/2}}du = \frac{\sqrt{\pi}\,\Gamma(s)}{\Gamma(s+1/2)},$$

by a direct calculation using the definition of the Gamma function. For $n \neq 0$ we can replace n by $|n|$, and then the inner integral can be computed using the representation

$$K_s(x) = \frac{2^{s-1}\Gamma(s+1/2)}{\sqrt{\pi}\,x^s}\int_{-\infty}^{\infty}\frac{\cos(xu)}{(u^2+1)^{s+1/2}}du,$$

which is valid for $\mathrm{Re}(s) > -1/2$ and $x > 0$ (see [AS64, 9.6.25]). Note that $K_s = K_{-s}$.

The asymptotics $K_0(z) \sim -\log(z)$ and $K_s(z) \sim \frac{1}{2}\Gamma(s)(\frac{1}{2}z)^{-s}$ for $\mathrm{Re}(s) > 0$ fixed as $z \to 0$ (see [AS64, 9.6.8, 9.6.9]) show that the integral in the series is holomorphic for $\mathrm{Re}(s) > -1$. This completes the proof. $\square$

Proposition 5.2.2. *Let $f \in H_{1/2,\tilde{\rho}_L^*}$. For $y \gg 0$ sufficiently large, the Borcherds lift of f has the Fourier expansion*

$$\begin{aligned}\Phi_{\Delta,r}(f,z) = &-4\sum_{m=1}^{\infty} c_f^+(|\Delta|m^2/4N, rm)\sum_{b(\Delta)}\left(\frac{\Delta}{b}\right)\log|1-e(mz+b/\Delta)| \\ &+2\sum_{m=1}^{\infty} c_f^-(|\Delta|m^2/4N, rm)\left(\frac{|\Delta|m^2}{4N}\right)^{-1/2}\sum_{b(\Delta)}\left(\frac{\Delta}{b}\right)\mathcal{F}(mz+b/\Delta) \\ &+\begin{cases}\sqrt{N}y(f,\theta_{1/2})^{\mathrm{reg}} - c_f^+(0,0)\left(\log(4\pi N y^2)+\Gamma'(1)\right) & \\ \qquad -\sqrt{N}y\, c_f^-(0,0)\left(\log(4\pi)-\log(Ny^2)+\Gamma'(1)\right) & \text{if } \Delta = 1, \\ 2\sqrt{\Delta}\, L(1,\chi_\Delta)\left(c_f^+(0,0)+\sqrt{N}y\, c_f^-(0,0)\right) & \text{if } \Delta > 1, \\ 0 & \text{if } \Delta < 0.\end{cases}\end{aligned}$$

Here the function $\mathcal{F}(z) : \mathbb{H} \to \mathbb{R}$ is defined by

$$\mathcal{F}(z) = \lim_{s\to 0}\left(\sum_{\ell\in\mathbb{Z}}\arcsin_s\left(\frac{y}{\sqrt{(x+\ell)^2+y^2}}\right) - y\frac{\sqrt{\pi}\,\Gamma(s)}{\Gamma(s+1/2)}\right),$$

compare Lemma 5.2.1.

Remark 5.2.3.
1. The singularities of $\Phi_{\Delta,r}(f,z)$ at Heegner points and geodesics given by semi-circles centered at the real line are not reproduced in the Fourier expansion above, but the part involving the function $\mathcal{F}$ captures the singularities along vertical geodesics.

2. By Dirichlet's class number formula we have

$$L(1,\chi_\Delta) = \frac{1}{\sqrt{\Delta}}h(\Delta)\log(\varepsilon_\Delta) = \frac{1}{2}\operatorname{tr}_1(-\Delta/4)$$

for $\Delta > 1$, where $h(\Delta)$ is the narrow class number of $\mathbb{Q}(\sqrt{\Delta})$, ε_Δ is the smallest unit > 1 of norm 1, and $\operatorname{tr}_1(-\Delta/4)$ is the Δ-th trace of the constant 1 function.

Proof of Proposition 5.2.2. The proof follows the arguments of [BO10b], Theorem 5.3. First, by [BO10b], Theorem 4.8, we can write

$$\begin{aligned}v^{-1/2}\overline{\Theta_{\Delta,r}(\tau,z)} = &\,\delta_{\Delta=1}\frac{\sqrt{N}y}{\sqrt{|\Delta|}}\theta_{1/2}(\tau) \\ &+\frac{\sqrt{N}y}{\sqrt{|\Delta|}}\sum_{n\geq 1}\sum_{M\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}}\left[\exp\left(-\frac{\pi n^2 N y^2}{|\Delta|v}\right)\Xi(\tau,\mu,n,0)\right]\Big|_{1/2,\tilde{\rho}_L^*} M,\end{aligned}$$

where $\mu = \left(\begin{smallmatrix} x & -x^2 \\ -1 & -x \end{smallmatrix}\right)$ and

$$\Xi(\tau, \mu, n, 0) = \left(\frac{\Delta}{n}\right) \overline{\varepsilon} \sqrt{|\Delta|} \sum_{h \in K'/K} \sum_{\substack{X \in K + rh \\ Q(X) \equiv \Delta Q(h)\,(\Delta)}} e\left(-Q_\Delta(X)\tau + n(X,\mu)_\Delta\right) \mathfrak{e}_h,$$

with $\varepsilon = 1$ if $\Delta > 0$ and $\varepsilon = i$ if $\Delta < 0$. Further, K denotes the one-dimensional negative definite sublattice

$$K = \left\{ \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} : b \in \mathbb{Z} \right\}$$

of L. Its dual lattice is given by

$$K' = \left\{ \begin{pmatrix} b/2N & 0 \\ 0 & -b/2N \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Inserting this into the definition of the theta lift, the unfolding argument yields

$$\Phi_{\Delta,r}(f,z) = \delta_{\Delta=1} \frac{\sqrt{N}y}{\sqrt{|\Delta|}} (f, \theta_{1/2})^{\mathrm{reg}} + \mathrm{CT}_{s=0}\, \Phi^0_{\Delta,r}(f,z,s),$$

where

$$\Phi^0_{\Delta,r}(f,z,s) = \frac{2\sqrt{N}y}{\sqrt{|\Delta|}} \sum_{n \geq 1} \int_{v=0}^{\infty} \int_{u=0}^{1} \exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v}\right) \langle f, \Xi(\tau,\mu,n,0)\rangle du \frac{dv}{v^{s+3/2}}.$$

The unfolding is justified for $y \gg 0$ by the same arguments as in [Bor98], Theorem 7.1. Let us write

$$f(\tau) = \sum_{h \in L'/L} \sum_{n \in \mathbb{Q}} c_f(n,h,v) e(n\tau) \mathfrak{e}_h$$

for the Fourier expansion of f for the moment. Since Δ is fundamental, the conditions $X \in K + rh$ and $Q(X) \equiv \Delta Q(h) \bmod \Delta$ are equivalent to $X = \Delta X'$ and $rX' \in K + h$ for some $X' \in K'$. Plugging in the definition of $\Xi(\tau, n, \mu, 0)$, and evaluating the integral over u, we obtain

$$\begin{aligned} \Phi^0_{\Delta,r}(f,z,s) = 2\sqrt{N}y\varepsilon &\sum_{X \in K'} \sum_{n \geq 1} \left(\frac{\Delta}{n}\right) e\left(-\operatorname{sgn}(\Delta) n (X,\mu)\right) \\ &\times \int_{v=0}^{\infty} c_f(-|\Delta| Q(X), rX, v) \exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v} + 4\pi |\Delta| Q(X) v\right) \frac{dv}{v^{s+3/2}}. \end{aligned}$$

Now we use the explicit form of the Fourier coefficients of f. The summand for $X = 0$

in $\Phi^0_{\Delta,r}(f,z,s)$ is given by

$$
\begin{aligned}
&2\sqrt{N}y\varepsilon \sum_{n\geq 1}\left(\frac{\Delta}{n}\right)\int_{v=0}^{\infty}\left(c_f^+(0,0)+c_f^-(0,0)v^{1/2}\right)\exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v}\right)\frac{dv}{v^{s+3/2}}\\
&= 2\varepsilon\left(Ny^2\right)^{-s}\left(c_f^+(0,0)\left(\frac{\pi}{|\Delta|}\right)^{-s-1/2}\Gamma(s+1/2)L(2s+1,\chi_\Delta)\right.\\
&\qquad\left. + \sqrt{N}y\, c_f^-(0,0)\left(\frac{\pi}{|\Delta|}\right)^{-s}\Gamma(s)L(2s,\chi_\Delta)\right).
\end{aligned}
$$

For $\Delta < 0$ the harmonic Maass form f transforms with ρ_L, which implies that its zero component vanishes, so $c_f^{\pm}(0,0) = 0$. For $\Delta > 0$ the completed Dirichlet L-function

$$\Lambda(s,\chi_\Delta) = (\pi/\Delta)^{-s/2}\Gamma(s/2)L(s,\chi_\Delta)$$

satisfies the functional equation $\Lambda(1-s,\chi_\Delta) = \Lambda(s,\chi_\Delta)$. It is holomorphic at $s=1$ if $\Delta > 1$. Taking the constant term at $s=0$, we get the contribution in the large bracket in the proposition.

For $X \in K'$ with $X \neq 0$ we have $-|\Delta|Q(X) > 0$. We can write

$$c_f(n,h,v) = c_f^+(n,h) + c_f^-(n,h)\sqrt{v}\beta^c_{1/2}(-4\pi n v)$$

for $n > 0$. The contribution coming from the coefficients $c_f^+(n,h)$ can be computed as in [BO10b], Theorem 5.2, and yields the first line of the Fourier expansion. Plugging in the definition of $\beta^c_{1/2}(s)$, it remains to compute

$$
\begin{aligned}
&4\sqrt{N}y\varepsilon \sum_{\substack{X\in K'\\ X\neq 0}} c_f^-(-|\Delta|Q(X), rX)\sum_{n\geq 1}\left(\frac{\Delta}{n}\right) e\left(-\operatorname{sgn}(\Delta)n(X,\mu)\right) \qquad (5.2.4)\\
&\times \int_{v=0}^{\infty}\left(\int_{w=0}^{1}\exp(-4\pi|\Delta|Q(X)w^2 v)dw\right)\exp\left(-\frac{\pi n^2 N y^2}{|\Delta| v} + 4\pi|\Delta|Q(X)v\right)\frac{dv}{v}.
\end{aligned}
$$

If we change the order of integration, the inner integral can be computed in terms of the K-Bessel function by [EMOT54, (3.471.9)], giving

$$\int_{v=0}^{\infty}\exp\left(4\pi|\Delta|Q(X)(1-w^2)v - \frac{\pi n^2 N y^2}{|\Delta| v}\right)\frac{dv}{v} = 2K_0\left(2\pi y|n|\sqrt{-4NQ(X)(1-w^2)}\right).$$

Write $X = \left(\begin{smallmatrix} m/2N & 0\\ 0 & -m/2N\end{smallmatrix}\right) \in K'\setminus\{0\}$ with $m\in\mathbb{Z}, m\neq 0$. Then $-Q(X) = m^2/4N$ and

$-(X,\mu) = mx$. We use the evaluation of the Gauss sum

$$\sum_{b(\Delta)} \left(\frac{\Delta}{b}\right) e(bn/|\Delta|) = \varepsilon \left(\frac{\Delta}{n}\right) \sqrt{|\Delta|}. \tag{5.2.5}$$

Then the expression in (5.2.4) becomes

$$2\sum_{m=1}^{\infty} a_f(|\Delta|m^2/4N, rm) \left(\frac{|\Delta|m^2}{4N}\right)^{-1/2} \sum_{b(\Delta)} \left(\frac{\Delta}{b}\right)$$
$$\times\, 2my \sum_{n\neq 0} e\left(n(mx + b/\Delta)\right) \int_0^1 K_0\left(2\pi my|n|\sqrt{1-w^2}\right) dw.$$

By Lemma 5.2.1 the second line agrees with $\mathcal{F}(mz + b/\Delta)$, which finishes the proof. $\square$

If the coefficients $c_f^+(n,h)$ vanish for $n<0$, then $\Phi_{\Delta,r}(f,z)$ does not have singularities at Heegner points, and extends to a continuous function on $\mathbb{H}$ which is not differentiable along the geodesics in $H_{\Delta,r}^-(f)$. Further, the estimates from Theorem 2.3.22 show that the coefficients $c_f^+(n,h)$ grow polynomially as $n\to\infty$, which implies that the series in the first line of the Fourier expansion given above converges on $\mathbb{H}$. In this case, we can derive the Fourier expansion of $\Phi_{\Delta,r}(f,z)$ on $\mathbb{H}\setminus H_{\Delta,r}^-(f)$, without assuming $y \gg 0$ to be large enough.

Corollary 5.2.4. *Let $f \in H_{1/2,\tilde{\rho}_L^*}$, and suppose that $c_f^+(n,h) = 0$ for all $n<0$ and $h \in L'/L$. Then the Fourier expansion of the Borcherds lift $\Phi_{\Delta,r}(f,z)$ on $\mathbb{H}\setminus H_{\Delta,r}^-(f)$ is given by the formula from Proposition 5.2.2 plus the expression*

$$-2\sum_{h\in L'/L}\sum_{n>0} c_f^-(n,h)n^{-1/2} \sum_{\substack{X\in L_{-|\Delta|n,rh}\\ a\neq 0}} \chi_\Delta(X)\mathbf{1}_X(z)\left(\arctan\left(\frac{\sqrt{4|\Delta|n}}{-\operatorname{sgn}(a)p_X(z)}\right) + \frac{\pi}{2}\right), \tag{5.2.6}$$

where $\mathbf{1}_X(z)$ denotes the characteristic function of the bounded component of $\mathbb{H}\setminus c_X$.

Remark 5.2.5. 1. Recall that for $X = \begin{pmatrix} -b/2N & -c/N \\ a & b/2N \end{pmatrix} \in L'$ we have

$$Q_\Delta(X_z) = \frac{1}{4}p_X^2(z), \qquad p_X(z) = -\frac{aN|z|^2 + bx + c}{y\sqrt{N}}.$$

Further, if $a\neq 0$ then a point z lies inside the bounded component of $\mathbb{H}\setminus c_X$ if and only if $\operatorname{sgn}(a)p_X(z) > 0$.

2. The sum in (5.2.6) is locally finite since for fixed n each point z lies in the bounded component of $\mathbb{H} \setminus c_X$ for only finitely many $X \in L_{-|\Delta|n,rh}$ with $a \neq 0$.

Proof. Let $\tilde{\Phi}_{\Delta,r}(f,z)$ denote $\Phi_{\Delta,r}(f,z)$ minus the expression in (5.2.6). Then we have $\tilde{\Phi}_{\Delta,r}(f,z) = \Phi_{\Delta,r}(f,z)$ for $y \gg 0$ large enough since the imaginary parts of points lying on geodesics c_X for $X \in L_{-|\Delta|n,rh}$ with $a \neq 0$ are bounded by a constant depending on n, and the sum over n is finite.

Further, for $a \neq 0$ and $z \notin c_X$ we can write

$$\begin{aligned}
&-2 \cdot \mathbf{1}_X(z) \left(\arctan\left(\frac{\sqrt{4|\Delta|n}}{-\operatorname{sgn}(a) p_X(z)} \right) + \frac{\pi}{2} \right) \\
&= \arctan\left(\frac{\sqrt{4|\Delta|n}}{|p_X(z)|} \right) - \left(\arctan\left(\frac{\sqrt{4|\Delta|n}}{-\operatorname{sgn}(a) p_X(z)} \right) + \mathbf{1}_X(z)\pi \right) \\
&= \arctan\left(\sqrt{\frac{Q_\Delta(X)}{-Q_\Delta(X_z)}} \right) - \operatorname{arccot}\left(\frac{-\operatorname{sgn}(a) p_X(z)}{\sqrt{4|\Delta|n}} \right).
\end{aligned}$$

Using that the function arccot is real analytic at the origin, and the shape of the singularities of $\Phi_{\Delta,r}(f,z)$ determined in Theorem 5.1.1, we see that $\tilde{\Phi}_{\Delta,r}(f,z)$ is actually real analytic on $\mathbb{H} \setminus \{\text{vertical geodesics in } H^-_{\Delta,r}(f)\}$. Since the first two lines of the Fourier expansion in Proposition 5.2.2 are also real analytic on this domain by the estimates from Theorem 2.3.22 and agree with $\tilde{\Phi}_{\Delta,r}(f,z)$ for $y \gg 0$, they have to agree with $\tilde{\Phi}_{\Delta,r}(f,z)$ on $\mathbb{H} \setminus \{\text{vertical geodesics in } H^-_{\Delta,r}(f)\}$. This completes the proof. □

5.3 The derivative of the Borcherds lift

We consider the derivative

$$\Phi'_{\Delta,r}(f,z) = \frac{\partial}{\partial z} \Phi_{\Delta,r}(f,z)$$

of the Borcherds lift.

Theorem 5.3.1. *Let $f \in H_{1/2,\tilde{\rho}^*_L}$. The derivative $\Phi'_{\Delta,r}(f,z)$ of the Borcherds lift is harmonic on $\mathbb{H} \setminus (H^+_{\Delta,r}(f) \cup H^-_{\Delta,r}(f))$ and transforms like a modular form of weight 2 under $\Gamma_0(N)$. If $\Delta \neq 1$ or if $c^+_f(0,0) = 0$, then $\Phi'_{\Delta,r}(f,z)$ is holomorphic on its domain.*

At a point $z_0 \in H^+_{\Delta,r}(f) \cup H^-_{\Delta,r}(f)$ it has a singularity of type

$$i\sqrt{N} \sum_{h\in L'/L} \sum_{n<0} c_f^+(n,h) \sum_{\substack{X\in L_{-|\Delta|n,rh} \\ z_0=z_X}} \chi_\Delta(X) \frac{p_X(z)}{Q_X(z)}$$
$$+ i\sqrt{N|\Delta|} \sum_{h\in L'/L} \sum_{n>0} c_f^-(n,h) \sum_{\substack{X\in L_{-|\Delta|n,rh} \\ z_0\in c_X}} \chi_\Delta(X) \frac{\operatorname{sgn}(p_X(z))}{Q_X(z)}.$$

Proof. The analytic properties of $\Phi'_{\Delta,r}(f,z)$ follow from the Laplace equation in Theorem 5.1.1 and the formula
$$\Delta_0 = -4y^2 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}.$$
The transformation behaviour follows from the fact that $\frac{\partial}{\partial z}$ equals the raising operator R_0 up to a constant. The types of singularities of $\Phi'_{\Delta,r}(f,z)$ are given by the derivatives of the types of singularities of $\Phi_{\Delta,r}(f,z)$, which can be computed using the formulas (2.4.2). □

Remark 5.3.2. Let $X = \begin{pmatrix} -b/2N & -c/N \\ a & b/2N \end{pmatrix} \in L_{-|\Delta|n,rh}$. For $n < 0$ we have

$$Q_X(z) = aNz^2 + bz + c = 0$$

exactly for the Heegner point $z = z_X$. Hence $\Phi'_{\Delta,r}(f,z)$ has simple poles at the Heegner points in $H^+_{\Delta,r}(f)$. For $n > 0$ the sign of

$$p_X(z) = -\frac{aN|z|^2 + bx + c}{y\sqrt{N}}$$

changes if z crosses the geodesic c_X. This means that $\Phi'_{\Delta,r}(f,z)$ has jump singularities along the geodesics in $H^-_{\Delta,r}(f)$.

Proposition 5.3.3. *Let $f \in H_{1/2,\tilde\rho_L^*}$. For $y \gg 0$ sufficiently large we have the Fourier expansion*

$$\begin{aligned}
\Phi'_{\Delta,r}(f,z) &= 4\pi i\sqrt{|\Delta|}\bar\varepsilon \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{\Delta}{n/d}\right) d\, c_f^{+}\left(|\Delta|d^2/4N, rd\right)\right) e(nz) \\
&\quad + 2\sum_{m=1}^{\infty} c_f^{-}(|\Delta|m^2/4N, rm)\left(\frac{|\Delta|}{4N}\right)^{-1/2}\sum_{b(\Delta)}\left(\frac{\Delta}{b}\right)\mathcal{F}'(mz+b/\Delta) \\
&\quad + \begin{cases} -\frac{i\sqrt{N}}{2}(f,\theta_{1/2})^{\mathrm{reg}} + \frac{i}{y}c_f^{+}(0,0) \\ \quad + \frac{i}{2}\sqrt{N}\, c_f^{-}(0,0)\left(\log(4\pi) - \log(Ny^2) - 2 + \Gamma'(1)\right) & \text{if } \Delta = 1, \\ -i\sqrt{N\Delta}\, L(1,\chi_\Delta) c_f^{-}(0,0) & \text{if } \Delta > 1, \\ 0 & \text{if } \Delta < 0. \end{cases}
\end{aligned}$$

where $\varepsilon = 1$ if $\Delta > 0$ if $\varepsilon = i$ for $\Delta < 0$, and

$$\mathcal{F}'(z) = -\frac{i}{2}\lim_{s\to 0}\left(y^{2s}\Gamma(s+1)\sum_{\ell\in\mathbb{Z}}\frac{\operatorname{sgn}(x+\ell)(\bar z+\ell)}{|z+\ell|^{2s+2}} - \Gamma(s)\right).$$

Proof. The derivative of $\mathcal{F}(z)$ can be computed most easily using the representation (5.2.3) of $\arcsin_s$ as an incomplete beta function. Using the formula (5.2.5) for the Gauss sum, the calculation of the remaining derivatives is straightforward. □

Remark 5.3.4. We check the Laplace equation from Theorem 5.1.1 on the level of Fourier expansions, at least for $y \gg 0$ sufficiently large. Using $\Delta_0 = -4y^2\frac{\partial}{\bar z}\frac{\partial}{z}$, and applying $-4y^2\frac{\partial}{\partial\bar z}$ to the expansion of $\Phi'_{\Delta,r}(f,z)$ from Proposition 5.3.3, we compute

$$\begin{aligned}
\Delta_0\Phi_{\Delta,r}(f,z) &= -16y^2\frac{\sqrt{N}}{\sqrt{|\Delta|}}\sum_{m=1}^{\infty} m c_f^{-}(|\Delta|m^2/4N, rm)\sum_{b(\Delta)}\left(\frac{\Delta}{b}\right)\left(\frac{\partial}{\partial\bar z}\mathcal{F}'\right)(mz+b/\Delta) \\
&\quad - \delta_{\Delta=1}\left(2c_f^{+}(0,0) + 2y\sqrt{N}c_f^{-}(0,0)\right),
\end{aligned}$$

where

$$\frac{\partial}{\partial\bar z}\mathcal{F}'(z) = \frac{1}{2}\lim_{s\to 0}\left(sy^{2s-1}\Gamma(s+1)\sum_{\ell\in\mathbb{Z}}\frac{\operatorname{sgn}(x+\ell)(x+\ell)}{|z+\ell|^{2s+2}}\right) = \frac{1}{2y}.$$

Here the residue of the series over ℓ can be computed from its Fourier expansion by similar arguments as in Lemma 5.2.1. If $\Delta \neq 1$, we obtain $\Delta_0\Phi_{\Delta,r}(f,z) = 0$ since the sum over the values of the nontrivial character $\left(\frac{\Delta}{\cdot}\right)$ vanishes, and if $\Delta = 1$ we get

$$\Delta_0 \Phi_{1,r}(f,z) = -2c_f^+(0,0) - 4y\sqrt{N}\left(\frac{1}{2}c_f^-(0,0) + 2\sum_{m=1}^{\infty} m c_f^-(m^2/4N, rm)\right).$$

The expression in the brackets on the right-hand side vanishes since it is the complex conjugate of the residue of the meromorphic 1-form $\sum_{h\in L'/L}(\xi_{1/2}f)_{rh}(\tau)\cdot\theta_h(\tau)\,d\tau$. Hence we obtain $\Delta_0\Phi_{1,r}(f,z) = -2c_f^+(0,0)$, in accordance with Theorem 5.1.1.

Again, we consider the special case that $c_f^+(n,h) = 0$ for all $n < 0$.

Corollary 5.3.5. *Let $f \in H_{1/2,\tilde{\rho}_L^*}$, and suppose that $c_f^+(n,h) = 0$ for all $n < 0$ and $h \in L'/L$. Then the Fourier expansion of the derivative $\Phi'_{\Delta,r}(f,z)$ of the Borcherds lift on $\mathbb{H} \setminus H_{\Delta,r}^-(f)$ is given by the formula from Proposition 5.3.3 plus the expression*

$$-2i\sqrt{|\Delta|N}\sum_{h\in L'/L}\sum_{n>0} c_f^-(n,h) \sum_{\substack{X\in L_{-|\Delta|n,rh}\\ a\neq 0}} \chi_\Delta(X)\frac{\mathbf{1}_X(z)\operatorname{sgn}(a)}{Q_X(z)},$$

where $\mathbf{1}_X(z)$ denotes the characteristic function of the bounded component of $\mathbb{H}\setminus c_X$.

Proof. This can either be proved by similar arguments as in the proof of Corollary 5.2.4, or by computing the derivative of the expression (5.2.6), using the formulas (2.4.2). □

5.4 Modular integrals with rational period functions and Borcherds products

For simplicity, we assume in this section that N is square free. Then the cusps of $\Gamma_0(N)$ can be represented by the fractions $1/c$ with $c \mid N$. Note that ∞ corresponds to $1/N$. The width of $1/c$ is given by $\alpha_{1/c} = N/c$. We choose the matrix $\sigma_{1/c} \in \mathrm{SL}_2(\mathbb{Z})$ sending ∞ to $1/c$ in the form

$$\sigma_{1/c} = \begin{pmatrix} 1 & \beta \\ c & N\gamma/c \end{pmatrix}$$

where $\beta, \gamma \in \mathbb{Z}$ are such that $N\gamma/c - c\beta = 1$. Then we can take the Atkin-Lehner involution corresponding to N/c as

$$W_{N/c} = \sigma_{1/c}\begin{pmatrix} N/c & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that $W_{N/c}\infty = 1/c$, so the Atkin-Lehner involutions act transitively on the cusps. Further, the expansion at the cusp $1/c$ of a function F, which is modular of weight $k \in \mathbb{Z}$,

is given by
$$(F|_k\sigma_{1/c})(z) = (c/N)^{k/2} \cdot (F|_k W_{N/c})(cz/N).$$
Since
$$\Phi_{\Delta,r}(f,z)|_0 W_{N/c} = \Phi_{\Delta,r}(f^{w_{N/c}}, z)$$
and consequently
$$\Phi'_{\Delta,r}(f,z)|_2 W_{N/c} = \Phi'_{\Delta,r}(f^{w_{N/c}}, z),$$
the expansion of $\Phi'_{\Delta,r}(f,z)$ at the cusp $1/c$ is essentially given by $\Phi'_{\Delta,r}(f^{w_{N/c}}, z)$.

5.4.1 Modular integrals with rational period functions

As an application of our extension of the Borcherds lift, we construct modular integrals of weight 2 for $\Gamma_0(N)$ with rational period functions from harmonic Maass forms of weight 1/2. Following Knopp [Kno74], we call a holomorphic function $F : \mathbb{H} \to \mathbb{C}$ a modular integral of weight $k \in \mathbb{Z}$ for $\Gamma_0(N)$ with rational period functions if

$$q_M(z) = F(z) - (F|_k M)(z)$$

is a rational function of z for each $M \in \Gamma_0(N)$, and if F is holomorphic at the cusps of $\Gamma_0(N)$, in the sense that $\lim_{y\to\infty}(F|_k M)(z)$ exists for every $M \in \mathrm{SL}_2(\mathbb{Z})$. Then the map $M \mapsto q_M$ defines a weight k cocycle for $\Gamma_0(N)$ with values in the rational functions which are holomorphic on $\mathbb{H}$, i.e., it satisfies

$$q_{MM'} = q_M|_k M' + q_{M'}$$

for all $M, M' \in \Gamma_0(N)$. Conversely, it follows from a more general result of Knopp [Kno74] that every such cocycle admits a holomorphic modular integral. Knopp's modular integrals are Poincaré series built from the cocycles. It was shown in [DIT10] and [DIT11] that certain generating series of (traces of) cycle integrals of weakly holomorphic modular functions for $\mathrm{SL}_2(\mathbb{Z})$ are modular integrals of weight 2 with rational period functions. Using the Borcherds lift we generalize their construction to higher level.

Proposition 5.4.1. *Let $\Delta \neq 1$ be a fundamental discriminant. Let $f \in H_{1/2,\tilde{\rho}_L^*}$ with $c_f^+(n,h) = 0$ for all $n < 0$ and $h \in L'/L$. Further, assume that $c_f^-(|\Delta|m^2/4N, rm) = 0$ for all $m \in \mathbb{Z}, m > 0$. Then the function*

$$F_{\Delta,r}(f,z) = -\frac{1}{4\pi}L(1,\chi_\Delta)c_f^-(0,0) + \frac{\bar{\varepsilon}}{\sqrt{N}}\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{\Delta}{n/d}\right) d\, c_f^+\left(|\Delta|d^2/4N, rd\right)\right) e(nz)$$

is holomorphic on $\mathbb{H}$ *and at the cusps of* $\Gamma_0(N)$*, and satisfies the transformation rule*

$$F_{\Delta,r}(f,z)|_2 M - F_{\Delta,r}(f,z) = -\frac{1}{\pi}\sum_{h\in L'/L}\sum_{n>0} c_f^-(n,h)\sum_{\substack{X\in L_{-|\Delta|n,rh}\\ a_{MX}<0<a_X}}\frac{\chi_\Delta(X)}{Q_X(z)}$$

for all $M\in\Gamma_0(N)$*, where* a_X *denotes the* a *entry of* X*. In particular,* $F_{\Delta,r}(f,z)$ *is a modular integral of weight* 2 *for* $\Gamma_0(N)$*.*

Remark 5.4.2.

1. The requirement $c_f^-(|\Delta|m^2/4N, rm) = 0$ for all $m\in\mathbb{Z}, m>0$, ensures that $\Phi'_{\Delta,r}(f,z)$ does not have singularities along vertical geodesics, and implies that the second line of the Fourier expansion in Proposition 5.3.3 vanishes.

2. The proof of the transformation behaviour works for arbitrary positive integers N, but the assumption that N is square free is used to obtain the Fourier expansions of $\Phi'_{\Delta,r}(f,z)$ at different cusps via Atkin-Lehner operators. One could compute the expansion at a cusp ℓ by choosing an appropriate sublattice K_ℓ instead of K in Proposition 5.2.2 and modify the computation of the expansion at ∞ correspondingly. However, the above result is certainly true without the assumption that N is square free, but the computations become much more technical.

Proof of Proposition 5.4.1. Let $z\in\mathbb{H}\setminus H^-_{\Delta,r}(h)$, and let

$$F^*_{\Delta,r}(f,z) = -\frac{1}{2\pi}\sum_{h\in L'/L}\sum_{n>0} c_f^-(n,h)\sum_{\substack{X\in L_{-|\Delta|n,rh}\\ a\neq 0}}\chi_\Delta(X)\frac{\mathbf{1}_X(z)\operatorname{sgn}(a)}{Q_X(z)}.$$

By Corollary 5.3.5 we have

$$\Phi'_{\Delta,r}(f,z) = 4\pi i\sqrt{N|\Delta|}(F_{\Delta,r}(f,z)+F^*_{\Delta,r}(f,z)).$$

Since $\Phi'_{\Delta,r}(f,z)$ transforms like a modular form of weight 2 for $\Gamma_0(N)$, we obtain

$$F_{\Delta,r}(f,z)|_2 M - F_{\Delta,r}(f,z)| = -F^*_{\Delta,r}(f,z)|_2 M + F^*_{\Delta,r}(f,z).$$

Using $Q_X(z)|_{-2}M = Q_{M^{-1}X}(z)$, we obtain that the right-hand side of the last formula equals

$$-\frac{1}{2\pi}\sum_{h\in L'/L}\sum_{n>0} c_f^-(n,h)\sum_{\substack{X\in L_{-|\Delta|n,rh}\\ a\neq 0}}\chi_\Delta(X)\frac{\mathbf{1}_X(z)\operatorname{sgn}(a_X)-\mathbf{1}_{MX}(Mz)\operatorname{sgn}(a_{MX})}{Q_X(z)}.$$

The characteristic functions $\mathbf{1}_X$ and $\mathbf{1}_{MX}$ are related by

$$\mathbf{1}_{MX}(Mz) = \begin{cases} \mathbf{1}_X(z), & \text{if } a_X \cdot a_{MX} > 0, \\ 1 - \mathbf{1}_X(z), & \text{if } a_X \cdot a_{MX} < 0. \end{cases}$$

In particular, all summands with $a_X \cdot a_{MX} > 0$ cancel out. In the remaining sum over X with $a_X \cdot a_{MX} < 0$, we replace X with $-X$ if $a_X < 0$, giving a factor 2. This proves the transformation behaviour of $F_{\Delta,r}(f,z)$ for $z \in \mathbb{H} \setminus H^-_{\Delta,r}(f)$. Since all the functions appearing in the transformation formula are holomorphic on $\mathbb{H}$, we obtain the transformation law by analytic continuation.

Using $\Phi'_{\Delta,r}(f,z)|_2 W_d = \Phi'_{\Delta,r}(f^{w_d}, z)$ we obtain

$$F_{\Delta,r}(f,z)|_2 W_d = F_{\Delta,r}(f^{w_d}, z) + F^*_{\Delta,r}(f^{w_d}, z) + F^*_{\Delta,r}(f,z)|W_d.$$

Since $F_{\Delta,r}(f^{w_d}, z)$ is holomorphic at ∞, and $F^*_{\Delta,r}(f^{w_d}, z)$ and $F^*_{\Delta,r}(f,z)|_2 W_d$ vanish as $y \to \infty$, we see that $F_{\Delta,r}(f,z)$ is holomorphic at the cusps. □

Example 5.4.3. Let $\Delta > 1$. We apply Proposition 5.4.1 to a harmonic Maass form $f \in H_{1/2,\rho^*_L}$ arising as the image of the regularized theta lift studied by Bruinier, Funke and Imamoglu in [BFI15] of a harmonic Maass form $F \in H^+_0(\Gamma_0(N))$. We assume that the constant coefficients $a^+_\ell(0)$ of F vanish at all cusps. By Theorem 4.1 in [BFI15] the Fourier expansion of the h-th component of f is given by

$$\begin{aligned} f_h(\tau) = &-2\,\mathrm{tr}_F(0,h)\sqrt{v} \\ &+ \sum_{n<0} \mathrm{tr}_F(-n,h)\sqrt{v}\beta_{1/2}(4\pi|n|v)e(n\tau) \\ &+ \sum_{n>0} \frac{\sqrt{N}}{\pi}\,\mathrm{tr}_F(-n,h)e(n\tau) \\ &+ \sum_{n>0} \mathrm{tr}^c_F(-n^2/4N,h)\sqrt{v}\beta^c_{1/2}(-4\pi n^2 v/4N)e(n^2\tau/4N), \end{aligned}$$

where

$$\mathrm{tr}_F(0,h) = -\delta_{0,h}\frac{1}{2\pi}\int^{\mathrm{reg}}_{\Gamma_0(N)\backslash\mathbb{H}} F(z)\frac{dx\,dy}{y^2}$$

is a regularized average value of F and

$$\mathrm{tr}^c_F(n,h) = \sum_{X \in \Gamma_0(N)\backslash L_{n,h}} \sum_{m<0} \left(a^+_{\ell_X}(m)e^{2\pi i\,\mathrm{Re}(c(X))m} + a^+_{\ell_{-X}}(m)e^{2\pi i\,\mathrm{Re}(c(-X))m} \right)$$

is a complementary trace, which differs by a sign from the complementary trace appearing in the Fourier expansion of the weight $k=0$ Millson lift in Theorem 3.3.1 since the Millson and the Bruinier-Funke-Imamoglu lift transform with different representations.

Our definition of the traces of cycle integrals equals $\frac{\pi}{\sqrt{N}}$ times the traces of cycle integrals defined in [BFI15], and the traces for $|m|/N$ being a square need to be regularized as explained in [BFI15], Section 3. Note that the trace of index 0 can be evaluated explicitly in terms of the principal parts of F at the cusps of $\Gamma_0(N)$, see [BF06], Remark 4.9, and that the complementary trace is nonzero only for finitely many n, see [BF06], Proposition 4.7. Observe that $c_f^+(n,h) = 0$ for $n < 0$ and $c_f^-(\Delta m^2/4N, rm) = 0$ for $m \in \mathbb{Z}, m > 0$, if $\Delta > 1$.

By Proposition 5.4.1, for $\Delta > 1$ a fundamental discriminant the function

$$F_{\Delta,r}(f,z) = \frac{1}{2\pi} L(1,\chi_\Delta)\,\mathrm{tr}_F(0,0) + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{\Delta}{n/d}\right) d\,\mathrm{tr}_F\left(\Delta d^2/4N, rd\right)\right) e(nz)$$

is a holomorphic function on $\mathbb{H}$, which transforms under the weight 2 slash operation of $M \in \Gamma_0(N)$ by

$$F_{\Delta,r}(f,z)|_2 M - F_{\Delta,r}(f,z) = -\frac{1}{\pi}\sum_{h\in L'/L}\sum_{n>0}\mathrm{tr}_F^c(-n^2/4N,h)\sum_{\substack{X\in L_{-\Delta n^2/4N,rh}\\ a_{MX}<0<a_X}}\frac{1}{Q_X(z)}.$$

Since $\chi_\Delta(X) = 1$ for $X \in L_{-\Delta n^2/4N,rh}$ we dropped it from the notation.

In the special case $N = 1$ and $F = J = j - 744$ (with $\mathrm{tr}_J(0,0) = 4$ and $\mathrm{tr}_J^c(-1/4,1) = 2$) we recover the result of Duke, Imamoglu and Tóth [DIT11] stated in the introduction.

5.4.2 Borcherds products

In this section we construct twisted Borcherds products of harmonic Maass forms $f \in H_{1/2,\tilde{\rho}_L^*}$. Let us first recall the definition and some properties of twisted Borcherds products of $f \in H^+_{1/2,\tilde{\rho}_L^*}$. For simplicity we assume $\Delta \neq 1$.

Theorem 5.4.4 ([BO10b], Theorem 6.1)**.** *Let $\Delta \neq 1$ be a fundamental discriminant, and let $f \in H^+_{1/2,\tilde{\rho}_L^*}$ be a harmonic Maass form with real coefficients $c_f^+(m,h)$ for all $m \in \mathbb{Q}$ and $h \in L'/L$, and assume that $c_f^+(m,h) \in \mathbb{Z}$ for $m \leq 0$. Then the infinite product*

$$\Psi_{\Delta,r}(f,z) = \prod_{m=1}^{\infty}\prod_{b(\Delta)}\left[1 - e(mz + b/\Delta)\right]^{\left(\frac{\Delta}{b}\right)c_f^+(|\Delta|m^2/4N,rm)}$$

converges for $y \gg 0$ sufficiently large and has a meromorphic continuation to all of $\mathbb{H}$. It is a meromorphic modular form of weight 0 for $\Gamma_0(N)$ and a unitary character, and its roots and poles lie on the CM points in $H^+_{\Delta,r}(f)$, their order being determined by the *coefficients* $c_f^+(m,h)$ *for* $m<0$. *Further, it is related to the Borcherds lift of* f *by*

$$\Phi_{\Delta,r}(f,z) = 2\sqrt{|\Delta|}c_f^+(0,0)L(1,\chi_\Delta) - 4\log|\Psi_{\Delta,r}(f,z)|. \tag{5.4.1}$$

In order to generalize the above Borcherds products to the full space $H_{1/2,\tilde{\rho}_L^*}$ we first recall the construction of certain weight 0 and weight 2 cocycles from [DIT17], which will appear in the transformation rule of the Borcherds product.

Lemma 5.4.5. *Let* $n>0$ *such that* $N|\Delta|n$ *is not a square, and let* $\mathcal{A}\in\Gamma_0(N)\backslash L_{-|\Delta|n,rh}$. *Then the function*

$$q_M^{\mathcal{A}}(z) = \sum_{\substack{X\in\mathcal{A}\\ a_{MX}<0<a_X}} \frac{1}{Q_X(z)}$$

defines a weight 2 *cocycle with values in the rational functions which are holomorphic on* $\mathbb{H}$.

Proof. As in the proof of Proposition 5.4.1 we compute

$$\sum_{\substack{X\in\mathcal{A}\\ a_{MX}<0<a_X}} \frac{1}{Q_X(z)} = \sum_{\substack{X\in\mathcal{A}\\ a>0}} \frac{\mathbf{1}_X(z)\operatorname{sgn}(a)}{Q_X(z)} - \sum_{\substack{X\in\mathcal{A}\\ a>0}} \frac{\mathbf{1}_X(z)\operatorname{sgn}(a)}{Q_X(z)}\bigg|_2 M$$

for z not lying on any geodesic c_X with $X\in\mathcal{A}$. This easily implies that the map $M\mapsto q_M^{\mathcal{A}}$ is a weight 2 cocycle. $\square$

Remark 5.4.6. We sketch a possible construction of a modular integral for $q_M^{\mathcal{A}}(z)$, which is due to Parson [Par93]. Since we will not use it in this work, we skip the details. The function

$$f_{\mathcal{A}}(z,s) = \sum_{Q\in\mathcal{A}} \frac{\operatorname{sgn}(a)}{Q_X(z)|Q_X(z)|^s}$$

converges for $\mathrm{Re}(s)>0$. By computing its Fourier expansion as in [Koh85], Proposition 2, we see that the limit $f_{\mathcal{A}}(z) = \lim_{s\to 0} f_{\mathcal{A}}(z,s)$ exists, and has an expansion of the form $f_{\mathcal{A}}(z) = a_0y^{-1} + \sum_{n\geq 1} a(n)q^n$ with $a(n)\ll n^\alpha$ for some $\alpha>0$. By subtracting a suitable multiple of the weight 2 non-holomorphic Eisenstein series we obtain a modular integral for $q_M^{\mathcal{A}}(z)$.

Next, we would like to construct a weight 0 cocycle $R_M^{\mathcal{A}}(z)$ with values in the holomorphic functions on $\mathbb{H}$ such that $\frac{\partial}{\partial z}R_M^{\mathcal{A}}(z) = q_M^{\mathcal{A}}(z)$. The following Proposition gives such a construction for general cocycles with values in rational functions which are holomorphic on $\mathbb{H}$.

Proposition 5.4.7 ([DIT17], Theorem 2.1). *Let* $F(z) = \sum_{n\geq 0} a(n)e(nz)$ *be a holomorphic modular integral of weight* 2 *for* $\Gamma_0(N)$ *with rational period functions* $q_M =$

$F|_2M - F$. Assume that $a(n) \ll n^\alpha$ for some $\alpha > 0$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $c \neq 0$ we let

$$\Lambda\left(s, \frac{a}{c}\right) = \left(\frac{2\pi}{c}\right)^{-s} \Gamma(s) \sum_{n \geq 1} a(n) e\left(\frac{an}{c}\right) n^{-s}$$

and

$$H\left(s, \frac{a}{c}\right) = \Lambda\left(s, \frac{a}{c}\right) + \int_1^\infty q_M(-d/c + it/c) t^{1-s} dt + \frac{a(0)}{s} - \frac{a(0)}{2-s}.$$

Then $H\left(s, \frac{a}{c}\right)$ is entire and satisfies the functional equation $H\left(s, \frac{a}{c}\right) = -H\left(2-s, -\frac{d}{c}\right)$. Further, for $c \neq 0$ we set

$$R_M(z) = -\frac{i}{c} H\left(1, \frac{a}{c}\right) + \int_{-\frac{d}{c}+\frac{i}{c}}^{z} q_M(w) dw + a(0)\frac{a+d}{c},$$

and for $M = \pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ we let $R_M(z) = na(0)$. Then $R_M(z)$ defines a weight 0 cocycle for $\Gamma_0(N)$ with values in the holomorphic functions on $\mathbb{H}$, and which satisfies $\frac{\partial}{\partial z} R_M(z) = q_M(z)$ for every $M \in \Gamma_0(N)$.

Proof. The proof is exactly the same as that of [DIT17], Theorem 2.1, so we only give a sketch. By a standard computation we obtain for $c \neq 0$ the integral representation

$$H\left(s, \frac{a}{c}\right) = -\int_1^\infty (F(z_{1/t}) - a(0)) t^{1-s} dt + \int_1^\infty (F(Mz_t) - a(0)) t^{s-1} dt,$$

where $z_t = -\frac{d}{c} + \frac{i}{ct}$. Since $z_{1/t} = -\frac{d}{c} + \frac{it}{c}$ and $Mz_t = \frac{a}{c} + \frac{it}{c}$, we see that $H\left(s, \frac{a}{c}\right)$ is entire and satisfies the claimed functional equation. Further, we let

$$G(z) = a(0)z + \sum_{n \geq 1} \frac{a(n)}{2\pi i n} e(nz)$$

be a primitive of $F(z)$. By taking the limit $s \to 1$ in $H\left(s, \frac{a}{c}\right)$ we obtain after a short calculation

$$R_M(z) = G(Mz) - G(z),$$

which is valid for all $M \in \Gamma_0(N)$ and defines a weight 0 cocycle with values in the holomorphic functions on $\mathbb{H}$, and $\frac{\partial}{\partial z} R_M(z) = q_M(z)$. $\square$

Lemma 5.4.8. *Let $q_M^{\mathcal{A}}$ be the weight 2 cocycle associated to $\mathcal{A} \in \Gamma_0(N) \setminus L_{-|\Delta|n,rh}$ as above. For $X \in \mathcal{A}$ let $w_X > w_X'$ denote the two real endpoints of the geodesic c_X. Let $F(z) = \sum_{n \geq 0} a(n) q^n$ be a modular integral for $q_M^{\mathcal{A}}$ with $a(n) \ll n^\alpha$ for some $\alpha > 0$ and* *let* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$*. Further, for* $c \neq 0$ *let*

$$L_F\left(s, \frac{a}{c}\right) = \sum_{n \geq 1} a(n) e\left(\frac{an}{c}\right) n^{-s}.$$

and

$$R_M^{\mathcal{A}}(z) = \frac{1}{\sqrt{4N|\Delta|n}} \sum_{\substack{X \in \mathcal{A} \\ a_{MX} < 0 < a_X}} (\log(z - w_X) - \log(z - w_X')) + \frac{1}{2\pi i} L_F\left(1, \frac{a}{c}\right) + a(0) \frac{a+d}{c},$$

and for $M = \pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ *we let* $R_M^{\mathcal{A}}(z) = na(0)$*. Then* $R_M^{\mathcal{A}}(z)$ *is a weight* 0 *cocycle with values in the holomorphic functions on* $\mathbb{H}$ *which satisfies* $\frac{\partial}{\partial z} R_M^{\mathcal{A}}(z) = q_M^{\mathcal{A}}(z)$*.*

Proof. Note that

$$q_M^{\mathcal{A}}(z) = \frac{1}{\sqrt{4N|\Delta|n}} \sum_{\substack{X \in \mathcal{A} \\ a_{MX} < 0 < a_X}} \left(\frac{1}{z - w_X} - \frac{1}{z - w_X'}\right).$$

Thus if we choose

$$\frac{1}{\sqrt{4N|\Delta|n}} \sum_{\substack{X \in \mathcal{A} \\ a_{MX} < 0 < a_X}} (\log(z - w_X) - \log(z - w_X'))$$

as a primitive for $q_M^{\mathcal{A}}(z)$, the formula for $R_M^{\mathcal{A}}(z)$ follows from Proposition 5.4.7. $\square$

Example 5.4.9. Let $N = 1, \Delta > 1$, and $M = S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We have

$$q_S^{\mathcal{A}}(z) = \sum_{\substack{X \in \mathcal{A} \\ c < 0 < a}} \frac{1}{Q_X(z)}.$$

It easily follows from the definition and the functional equation of $H(s, 0)$ given in Proposition 5.4.7 that

$$L_F(1, 0) = -\frac{2\pi i}{\sqrt{4\Delta n}} \sum_{\substack{X \in \mathcal{A} \\ c < 0 < a}} (\log(i - w_X) - \log(i - w_X'))$$

independently of the modular integral F for $q^{\mathcal{A}}$. In particular, we obtain

$$R_S^{\mathcal{A}}(z) = \frac{1}{\sqrt{4\Delta n}} \sum_{\substack{X \in \mathcal{A} \\ c < 0 < a}} \left(\log\left(\frac{z - w_X}{i - w_X}\right) - \log\left(\frac{z - w_X'}{i - w_X'}\right)\right).$$

We can now state the transformation behaviour of the Borcherds product associated to $f \in H_{1/2,\tilde{\rho}_L^*}$.

Theorem 5.4.10. *Let $\Delta \neq 1$ be a fundamental discriminant. Let $f \in H_{1/2,\tilde{\rho}_L^*}$ and suppose that $c_f^+(|\Delta|m^2/4N, rm) \in \mathbb{R}$ for all $m \in \mathbb{Z}, m > 0$. Further, assume that $c_f^+(n,h) = 0$ for all $n < 0, h \in L'/L$, and that $c_f^-(|\Delta|m^2/4N, rm) = 0$ for all $m \in \mathbb{Z}, m > 0$. Then the infinite product*

$$\Psi_{\Delta,r}(f,z) = \prod_{m=1}^{\infty} \prod_{b(\Delta)} [1 - e(mz + b/\Delta)]^{\left(\frac{\Delta}{b}\right) c_f^+(|\Delta|m^2/4N, rm)} \tag{5.4.2}$$

$$\times e\left(\frac{\sqrt{|\Delta|N}}{4\pi} L(1,\chi_\Delta) c_f^-(0,0) z\right) \tag{5.4.3}$$

converges to a holomorphic function on $\mathbb{H}$ transforming as

$$\Psi_{\Delta,r}(f, Mz) = \chi(M) \mu_{\Delta,r}(f, M, z) \Psi_{\Delta,r}(f,z) \tag{5.4.4}$$

for all $M \in \Gamma_0(N)$, where χ is a character of $\Gamma_0(N)$ and

$$\mu_{\Delta,r}(f,M,z) = \prod_{h \in L'/L} \prod_{n>0} \prod_{\mathcal{A} \in \Gamma_0(N) \backslash L_{-|\Delta|n,rh}} e\left(-\frac{\sqrt{|\Delta|N}}{\pi} c_f^-(n,h) \chi_\Delta(\mathcal{A}) R_M^{\mathcal{A}}(z)\right),$$

where $R_M^{\mathcal{A}}(z)$ is the weight 0 cocycle with $\frac{\partial}{\partial z} R_M^{\mathcal{A}}(z) = q_M^{\mathcal{A}}(z)$. Further, its logarithmic derivative is given by

$$\frac{\partial}{\partial z} \log\left(\Psi_{\Delta,r}(f,z)\right) = -2\pi i \sqrt{|\Delta|N} F_{\Delta,r}(f,z),$$

where $F_{\Delta,r}(f,z)$ is the modular integral defined in Proposition 5.4.1.

Proof. Using Proposition 5.4.1 we see after a short calculation that the logarithmic derivatives of $\Psi_{\Delta,r}(f,Mz)$ and $\mu_{\Delta,r}(f,M,z)\Psi_{\Delta,r}(f,z)$ agree. Further, both functions are holomorphic and non-vanishing on $\mathbb{H}$. Hence they are constant multiples of each other. This proves the transformation behaviour.

The fact that $R_M^{\mathcal{A}}(z)$ is a weight 0 cocycle together with the transformation formula of the Borcherds product implies that χ is a character of $\Gamma_0(N)$. $\square$

Remark 5.4.11. Note that the function

$$\begin{aligned}\tilde{F}_{\Delta,r}(f,z) = &-\frac{1}{4\pi} L(1,\chi_\Delta) c_f^-(0,0) z \\ &- \frac{1}{2\pi i \sqrt{|\Delta|N}} \sum_{b(\Delta)} \left(\frac{\Delta}{b}\right) \sum_{m=1}^{\infty} c_f^+(|\Delta|m^2/4N, rm) \log[1 - e(mz + b/\Delta)]\end{aligned}$$

is holomorphic on $\mathbb{H}$ and satisfies $\frac{\partial}{\partial z}\tilde{F}_{\Delta,r}(f,z) = F_{\Delta,r}(f,z)$. Further, we have

$$\Psi_{\Delta,r}(f,z) = e\left(-\sqrt{|\Delta|N}\tilde{F}_{\Delta,r}(f,z)\right).$$

In view of these relations and Proposition 5.4.1 the above theorem is not surprising. However, we chose the formulation of the theorem to emphasize the analogy with the Borcherds products of harmonic Maass forms which map to cusp forms under $\xi_{1/2}$.

Example 5.4.12. Let $\Delta > 1$, and let $f \in H_{1/2,\rho_L^*}$ be the Bruinier-Funke-Imamoglu lift of a harmonic Maass form $F \in H_0^+(\Gamma_0(N))$ with vanishing constant coefficients $a_\ell^+(0)$ at all cusps as in Example 5.4.3. Its Borcherds lift is given by

$$\begin{aligned} \Psi_{\Delta,r}\left(\frac{\pi}{\sqrt{N}}f, z\right) &= \prod_{m=1}^{\infty}\prod_{b(\Delta)}[1 - e(mz + b/\Delta)]^{\left(\frac{\Delta}{b}\right)\operatorname{tr}_F(|\Delta|m^2/4N, rm)} \\ &\quad \times e\left(-\frac{\sqrt{\Delta}}{2}L(1,\chi_\Delta)\operatorname{tr}_F(0,0)z\right). \end{aligned}$$

For $N = 1$ and $F = J = j - 744$ (with $\operatorname{tr}_J(0,0) = 4$ and $\operatorname{tr}_J^c(-1/4,1) = 2$) we obtain the theorem in the introduction. Note that the relations $S^4 = 1, (ST)^6 = 1$ and $\chi(T) = 1$ imply that $\chi = 1$ for $N = 1$.

6 Outlook

The methods used in this work are applicable in many other situations. We describe some future projects which are related to the present work, and mention some open problems arising from this work.

Shintani theta lifts of harmonic Maass forms

As pointed out in Section 3.5, the Millson and the Shintani lift can be extended to the full spaces of harmonic Maass form $H_{-2k}(\Gamma)$ and $H_{2k+2}(\Gamma)$, respectively. The coefficients of the holomorphic part of the Shintani lift are given by regularized cycle integrals of harmonic Maass forms, which can also be understood as special values of regularized L-functions. As an application, one can construct $\xi_{3/2}$-preimages of Zagier's generating series of singular moduli f_d of weight $1/2$ (see [Zag02]) as the Shintani lift of a ξ_2-preimage $\tilde{J} \in H_2(\mathrm{SL}_2(\mathbb{Z}))$ of J, and use them to relate the regularized Petersson norm $(f_d, f_d)^{\mathrm{reg}}$ of f_d (see [BDE16]) to the central value of the d-th twist of the regularized L-function of $\tilde{J}$. This will be the topic of upcoming joint work with Claudia Alfes-Neumann.

Regularized Kudla-Millson lifts in genus 2

Let V be a rational quadratic space of signature (p, q) and let $1 \leq n \leq p$. In [KM86, KM90], Kudla and Millson constructed a special Schwarz function φ on V^n with values in the closed differential forms of degree nq on the symmetric space associated with $O(p, q)$, and they used the corresponding theta functions and lifts to prove that the generating series of period integrals of compactly supported differential forms along certain special cycles are Siegel cusp forms of weight $(p + q)/2$ of genus n. In [BF06], Bruinier and Funke extended this lift in signature $(1, 2)$ and genus 1 to the space $H_0^+(\Gamma)$ of harmonic weak Maass forms, thereby proving the weight $3/2$ modularity of generating series of traces of CM values of weight 0 harmonic Maass forms. In this case, the Kudla-Millson theta function is given by $\Theta_{KM}(\tau, z)d\mu(z)$. In signature $(2, 1)$, we can choose $n = 1$ or $n = 2$. For $n = 1$, the Kudla-Millson theta function is given by $\Theta_{Sh,0}(\tau, z)dz + \overline{\Theta_{Sh,0}(\tau, z)}d\bar{z}$, i.e., it gives rise to the Shintani theta lift, and for $n = 2$, the Kudla-Millson Schwartz form gives rise to a theta lift from weight 0 harmonic Maass forms to weight $3/2$ (non-holomorphic) Siegel modular forms of genus 2. The investigation of the latter lift is a joint project with Michalis Neururer. First computations have shown that the theta lift converges without regularization due to the very rapid decay of the theta function. Further, we expect the coefficients $a(Q)$ of the lift corresponding to positive definite quadratic forms Q to be related to **values** of the input function at the CM point corresponding to Q, rather than to **traces** of CM values of the input, which appear in the genus 1 case.

Borcherds lifts of harmonic Maass forms in signature $(n, 2)$

In the present work, we investigated the Borcherds lift in signature $(1, 2)$ (by identifying the Grassmannian with the upper half-plane, and modular forms for the orthogonal group with elliptic modular forms), and showed that it maps harmonic Maass forms of weight $1/2$ to real analytic modular functions with singularities at CM points and geodesics in $\mathbb{H}$. Let L be an even lattice in a rational quadratic space V of signature $(n, 2)$, let Γ be a congruence subgroup of $O(L)$ which fixes the classes of L'/L, and let D be the Grassmannian of negative definite planes in $V(\mathbb{R})$. In [Bor98] and [Bru02] the authors considered the Borcherds lift on weight $1 - n/2$ weakly holomorphic modular forms and harmonic Maass forms in $H^+_{1-n/2,\rho_L}$, and showed that it yields real analytic Γ-invariant functions on D with singularities at Heegner divisors, which can be thought of as embedded sub-Grasmmanians associated to $O(n-1, 2)$. Using the methods of this work we can also extend the Borcherds lift in signature $(n, 2)$ to a map from the full space $H_{1-n/2,\rho_L}$ to real analytic modular functions on D with singularities along embedded sub-Grassmannians associated to $O(n-1, 2)$ and $O(n, 1)$. We hope to come back to this problem in the near future.

Open Problems

Problem 1

The proof of the growth estimates for the holomorphic coefficients of a harmonic Maass form of weight $1/2$ whose holomorphic principal part vanishes (see Theorem 2.3.22) is extremely complicated, especially if we compare it to the simple proof (Hecke bound applied to $\xi_{1/2}f$) of the estimate for the non-holomorphic coefficients of a harmonic Maass form $f \in H^+_{1/2,\rho_L}$. For the applications in this work, any polynomial bound would be sufficient, and there should be a simpler proof of such a polynomial estimate.

Problem 2

It would be desirable to give a more explicit construction of the modular function F from Lemma 4.2.2. Since there is a lot of freedom in the requirements on F, e.g., the principal part at ∞ may have poles of arbitrary order, the author believes that there should be simple trick to find such a function F explicitly. This would be useful for the numerical computation of Petersson inner products of harmonic Maass forms with unary theta functions.

Problem 3

Numerical experiments suggest that the traces in Theorem 4.2.6 are integers already without the factors 6 and $6t$. Yingkun Li informed the author that he found a different construction of ξ-preimages of unary theta series which have holomorphic parts whose Fourier coefficients are rational numbers with bounded denominators. This also suggests that the factor t is not necessary.

Problem 4

The applications of the Millson theta lift presented in this work only use the lift for $k = 0$, i.e., the lift from weight 0 to weight 1/2 harmonic Maass forms. It would be interesting to find applications of the higher weight Millson lift of $F \in H^+_{-2k}(\Gamma)$, apart from proving the modularity of generating series of traces of CM values of $R^k_{-2k}F$ and traces of cycle integrals of $\xi_{-2k}F$.

Problem 5

In Lemma 2.3.17 we gave a basis of the space $M_{1/2,\rho_L^*}$ for the lattice $L = (\mathbb{Z}, -Nx^2)$, consisting of unary theta series, and thus resembling the Serre-Stark theorem. In Section 2.4.2 we defined unary theta series $\overline{\Theta_{\ell,0}(\tau)}$ of weight 1/2 corresponding to one-dimensional sublattices K_ℓ of an even lattice L of signature $(1,2)$, where ℓ is an isotropic vector of $L \otimes \mathbb{Q}$. Thus the question arises whether these theta series form a basis (or at least a generating system) for $M_{1/2,\rho_L^*}$. This problem could be adressed using an explicit description of $M_{1/2,\rho_L^*}$ in terms of invariants of the Weil representation given in [Sko08].

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